This book is a collection of research articles written by research scholars, postdoc fellows and faculty members who have been doing research in the field of Mathematics and Statistics. The book begins with six articles in algebra, then five articles in Number Theory. The remaining part of the book covers topics from Differential Equation, Functional Analysis, Game Theory and Statistics. The articles enunciate either some original research works or some recent developments in those areas.

This book is edited by Mr. Prabhat Dutta. He did his masters from Dibrugarh University. He has been teaching in Lakhimpur Girls'College since 1995.

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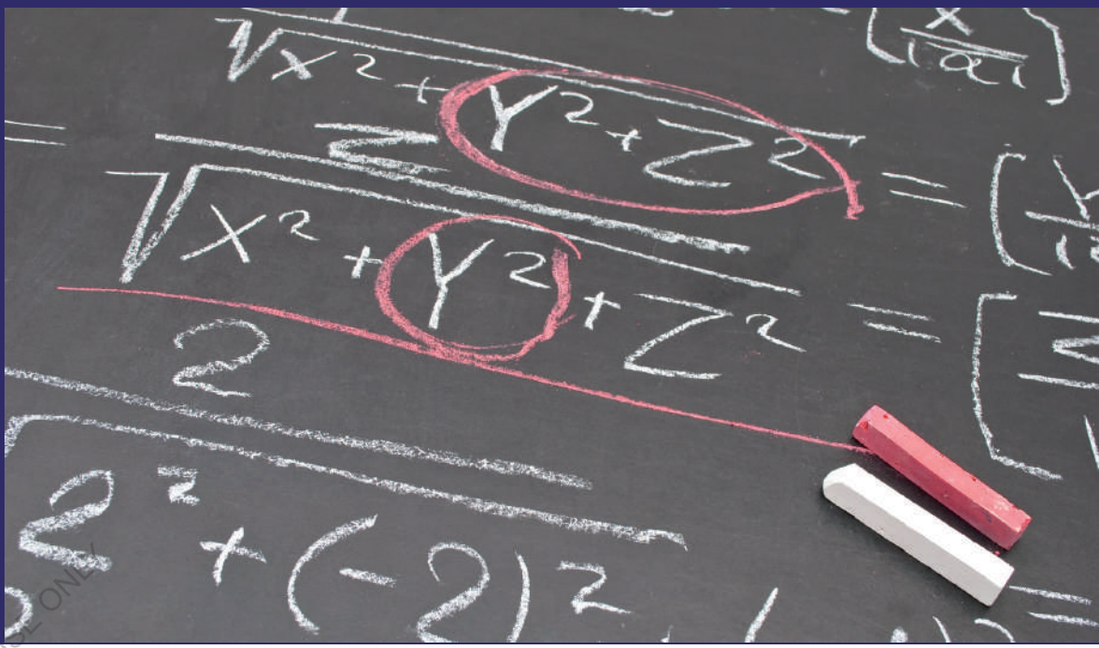
Recent Trends in Mathematical Sciences

Dutta



Prabhat Dutta

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Preface

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It is hoped that this book would serve as a ready reference for someone who is interested in the topics presented here. A generous sprinkling of open problems in almost all the articles makes it easy to look for research problems in these areas and the editor hopes that it will serve the mathematical community well.

Mr. Prabhat Dutta
Editor

Contents

On certain generalizations of commuting probability of a finite ring

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Abstract. For many years people have been studying the commuting probability of an algebraic structure its generalizations. MacHale initiated the study of commuting probability of a finite ring in the year 1976. After MacHale, many authors have studied this ratio and its generalizations. In this article, we give a brief survey on certain generalizations of the commuting probability of a finite ring and collect recent results on this notion.

2010 Mathematical Sciences Classification. 16U70.

Keywords. finite ring, commuting probability.

1 Introduction

Let F be a finite algebraic structure. The commuting probability of F is given by

$$\Pr(F) = \frac{|\{(x, y) \in F \times F : xy = yx\}|}{|F \times F|}.$$

That is $\Pr(F)$ is the probability that a randomly chosen pair of elements of F commute. Clearly, $\Pr(F) = 1$ if and only if F is commutative. The study of commuting probability was introduced by Erdős and Turán [8] in the year 1968 considering a finite group G. The commuting probability of a finite group G, denoted by $\Pr(G)$, is defined as

$$\Pr(G) = \frac{|\{(x, y) \in G \times G : xy = yx\}|}{|G \times G|}.$$

Following Erdős and Turán, many mathematicians have studied the ratio $\Pr(G)$ and its generalizations. They found certain computing formulae and bounds for the probability. They also characterized certain groups in term of this probabilities. A survey on the generalizations of $\Pr(G)$ can be found in [3].

The study of commuting probability of a finite ring was introduced by MacHale [9] in the year 1976. The commuting probability of a finite group R, denoted by $\Pr(R)$, is defined as

$$\Pr(R) = \frac{|\{(x, y) \in R \times R : xy = yx\}|}{|R \times R|}.$$

MacHale showed that for any finite ring R, $\Pr(R) \notin (\frac{5}{8}, 1)$. Unlike $\Pr(G)$, the research on $\Pr(R)$ was neglected for many years until MacHale resumed the study of commuting probability of finite rings together with Buckley and Shé in [1, 2]. This ratio was further studied by Dutta and Basnet in [5]. They obtained the following results on $\Pr(R)$.

Theorem 1.1. *[9, Theorem 1] If R is a finite non-commutative ring then $\Pr(R) \leq \frac{5}{8}$. The equality holds if and only if $|R : Z(R)| = 4$.*

Above theorem shows that there is no finite ring R such that $\Pr(R) \in (\frac{5}{8}, 1)$.

Theorem 1.2. *[4, Corollary 2.18] If R is a finite ring then*

$$\Pr(R) \geq \frac{1}{|[R, R]|}\left(1 + \frac{|[R, R]| - 1}{|S : Z(R)|}\right).$$

Theorem 1.3. *[9, Theorem 2] Let R be a non-commutative ring and p the smallest prime dividing order of R. Then*

$$\Pr(R) \leq \frac{p^2 + p - 1}{p^3}.$$

The equality holds if and only if $|R : Z(R)| = p^2$.

Theorem 1.4. *[9, Theorem 4] If S is a subring of a finite ring R then $\Pr(R) \leq \Pr(S)$.*

Theorem 1.5. *[5, Theorem 2.1(b)] Let R be a finite non-commutative ring. If p is the smallest prime dividing $|R|$ then*

$$\Pr(R) \leq \frac{(p-1)|Z(R)| + |R|}{p|R|}$$

with equality if and only if $|R : C_R(r)| = p$ for all $r \notin Z(R)$.

Theorem 1.6. *[5, Theorem 2.3] Let N be an ideal of a finite non-commutative ring R. Then*

$$\Pr(R) \leq \Pr(R/N) \Pr(N).$$

The equality holds if $N \cap [R, R] = \{0\}$.

In this article, we discuss certain generalization of commuting probability of finite rings and give a brief survey of the results obtained for these generalizations.

2 Relative commuting probability of a finite ring

Let S be a subring of a finite ring R. Dutta et al. [4] generalized $\Pr(R)$ through the following ratio

$$\Pr(S,R) := \frac{|\{(x,y) \in S \times R : xy = yx\}|}{|S \times R|}$$

where S is a subring of R. $\Pr(S,R)$ is called relative commuting probability of R with respect of the subring S. They obtained the following results on $\Pr(S,R)$.

Theorem 2.1. *[4, Theorem 2.5] Let S be a subring of a finite ring R and p be the smallest prime dividing $|R|$. Then*

$$\frac{|Z(S,R)|}{|S|} + \frac{p(|S| - |Z(S,R)|)}{|S||R|} \leq \Pr(S,R) \leq \frac{(p-1)|Z(S,R)| + |S|}{p|S|},$$

where $Z(S,R) := \{s \in S : sr = rs \text{ for all } r \in R\}$.

Theorem 2.2. *[4, Theorem 2.16] Let S be a subring of a finite ring R. Then*

$$\Pr(S,R) \geq \frac{1}{|K(S,R)|}\left(1 + \frac{|K(S,R)| - 1}{|S : Z(S,R)|}\right).$$

Theorem 2.3. *[4, Theorem 2.17] Let S be a subring of a finite ring R. Then*

$$\Pr(S,R) \geq \frac{1}{|[S,R]|}\left(1 + \frac{|[S,R]| - 1}{|S : Z(S,R)|}\right).$$

Theorem 2.4. *Let S be a commutative subring of a finite ring R such that $\Pr(S,R) = \frac{2p-1}{p^2}$, for some prime p. Then p divides $|R|$. Moreover, if p is the smallest prime dividing $|R|$, then*

$$\frac{S}{Z(S,R)} \cong \mathbb{Z}_p.$$

Theorem 2.5. *[4, Theorem 2.10] Let S be a non-commutative subring of a finite ring R such that $\Pr(S,R) = \frac{p^2+p-1}{p^3}$, for some prime p. Then p divides $|R|$. Moreover, if p is the smallest prime dividing $|R|$, then*

$$\frac{S}{Z(S,R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

Let $[R,R]$ be the subgroup of $(R,+)$ generated by all additive commutators of R and $Z(R) := \{x \in R : xy = yx \text{ for all } y \in R\}$. In [2], Buckley et al. introduced the notion of $\mathbb{Z}$-isoclinism of rings. Two rings R_1 and R_2 are said to be $\mathbb{Z}$-isoclinic if there exists an isomorphism ψ from the factor group $\frac{R_1}{Z(R_1)}$ to $\frac{R_2}{Z(R_2)}$, and an isomorphism β from $[R_1,R_1]$ to $[R_2,R_2]$ such that the following diagram commutes

$$\begin{array}{ccc} \frac{R_1}{Z(R_1)} \times \frac{R_1}{Z(R_1)} & \xrightarrow{\psi \times \psi} & \frac{R_1}{Z(R_1)} \times \frac{R_1}{Z(R_1)} \\ \downarrow & & \downarrow \\ [R_1,R_1] & \xrightarrow{\beta} & [R_2,R_2]. \end{array}$$

Buckley et al. [2] also proved that commuting probabilities of two $\mathbb{Z}$-isoclinic finite rings are same. Dutta et al. [4] further generalized the notion of $\mathbb{Z}$-isoclinism, given in the following definition.

Definition 2.6. *[4, Definition 5.1] Let R_1 and R_2 be two rings with subrings S_1 and S_2 respectively. A pair (S_1, R_1) is said to be $\mathbb{Z}$-isoclinic to (S_2, R_2) if there exist additive group isomorphisms $\alpha : \frac{R_1}{Z(S_1,R_1)} \to \frac{R_2}{Z(S_2,R_2)}$ such that $\alpha\left(\frac{S_1}{Z(S_1,R_1)}\right) = \frac{S_2}{Z(S_2,R_2)}$ and $\beta : [S_1, R_1] \to [S_2, R_2]$ such that $\beta([u_1, v_1]) = [u_2, v_2]$ whenever $u_i \in S_i$, $v_i \in R_i$ for $i = 1, 2$; $\alpha(u_1 + Z(S_1, R_1)) = u_2 + Z(S_2, R_2)$ and $\alpha(v_1 + Z(S_1, R_1)) = v_2 + Z(S_2, R_2)$. Such a pair of mappings (α, β) is called $\mathbb{Z}$-isoclinism between the pairs of rings (S_1, R_1) and (S_2, R_2).*

Recently, Dutta et al. [4] proved the following results.

Theorem 2.7. *[4, Theorem 3.3] Let R_1 and R_2 be two finite rings with subrings S_1 and S_2 respectively. If the pairs (S_1, R_1) and (S_2, R_2) are $\mathbb{Z}$-isoclinic then*

$$\Pr(S_1, R_1) = \Pr(S_2, R_2).$$

It is observed that the above results may be generalized for the generalized commuting probabilities of finite rings $\Pr_r(R), \Pr_r(S, R)$ and $\Pr_r(S, K)$ where S, K are additive subgroups of R and r is a given element of R.

Dutta and Nath [6] further generalized the notion of relative commuting probability of a ring R through the following ratio

$$\Pr_r(S, R) = \frac{|\{(x, y) \in S \times R : [x, y] = r\}|}{|S||R|}.$$

$\Pr_r(S, R)$ is called relative r-commuting probability of a finite ring R with respect to the subring S of R. They have obtained the following results.

Theorem 2.8. *[6, Proposition 1] Let S be a subring of a finite ring R and $r \in R$. Then $\Pr_r(S, R) = \Pr_{-r}(R, S)$. However, if $2r = 0$ then $\Pr_r(S, R) = \Pr_r(R, S)$.*

Theorem 2.9. *[6, Proposition 2] Let S_1 and S_2 be two subrings of the finite rings R_1 and R_2 respectively. If $(r_1, r_2) \in R_1 \times R_2$ then*

$$\Pr_{(r_1,r_2)}(S_1 \times S_2, R_1 \times R_2) = \Pr_{r_1}(S_1, R_1)\Pr_{r_2}(S_2, R_2).$$

Theorem 2.10. *[6, Proposition 3] Let S be a subring of a finite ring R. If p is the smallest prime dividing $|R|$ and $r \neq 0$ then*

$$\Pr_r(S, R) \leq \frac{|S| - |Z(S, R)|}{p|S|} < \frac{1}{p}.$$

Theorem 2.11. *[6, Proposition 4] Let S be a subring of a finite ring R. Then $\Pr_r(S, R) \leq \Pr(S, R)$ with equality if and only if $r = 0$.*

Theorem 2.12. *[6, Proposition 5] If $S_1 \subseteq S_2$ are two subrings of a finite ring R then*

$$\Pr_r(S_1, R) \leq |S_2 : S_1|\Pr_r(S_2, R).$$

Theorem 2.13. *[6, Corollary 3] If S is a subring of a finite ring R then*

$$\Pr_r(S,R) \leq |R:S|\Pr_r(R).$$

For any subring S of R, let $m_S = \min\{|[x,R]| : x \in S \setminus Z(S,R)\}$ and $M_S = \max\{|[x,R]| : x \in S \setminus Z(S,R)\}$. In the following theorem, we give bounds for $\Pr(S,R)$ in terms of m_S and M_S.

Theorem 2.14. *[6, Theorem 2] Let S be a subring of a finite ring R. Then*

$$\frac{1}{M_S}\left(1+\frac{M_S-1}{|S:Z(S,R)|}\right) \leq \Pr(S,R) \leq \frac{1}{m_S}\left(1+\frac{m_S-1}{|S:Z(S,R)|}\right).$$

The equality holds if and only if $m_S = M_S = |[x,R]|$ for all $x \in S \setminus Z(S,R)$.

It is worth mentioning here that the lower bound obtained in Theorem 2.14 is better than the lower bound given in Theorem 2.2 for $\Pr(S,R)$ and the upper bound obtained in Theorem 2.14 is better than the upper bound given in Theorem 1.2 for $\Pr(R)$.

3 Generalized commuting probability of a finite ring

Let S and K be two additive subgroups of R and $r \in R$. Dutta and Nath [7] defined $\Pr_r(S,K)$ in the following way

$$\Pr_r(S,K) = \frac{|\{(s,k) \in S \times K : [s,k] = r\}|}{|S \times K|}.$$

Thus $\Pr_r(S,K)$ is the probability that the additive commutator of a randomly chosen pair of elements, one from S and the other from K, is equal to a given element r of R. $\Pr_r(S,K)$ is called generalized r-commuting probability of R with respect to the subgroups S and K. If $r = 0$ then we write

$$\Pr_r(S,K) = \Pr(S,K) = \frac{|\{(s,k) \in S \times K : sk = ks\}|}{|S \times K|}.$$

Note that if S is a subring and $K = R$ then $\Pr(S,K)$ coincides with $\Pr(S,R)$.

Let $[S,K]$ and $[s,K]$ for $s \in S$ denote the additive subgroups of $(R,+)$ generated by the sets $\{[s,k] : s \in S, k \in K\}$ and $\{[s,k] : k \in K\}$ respectively. Let $Z(S,K) := \{s \in S : sk = ks \text{ for all } k \in K\}$. Dutta and Nath obtained the following computing formula for $\Pr_r(S,K)$.

Theorem 3.1. *[7, Theorem 2.1] Let S and K be two additive subgroups of R. Then*

$$\Pr_r(S,K) = \frac{1}{|S||K|}\sum_{\substack{s\in S\\ r\in[s,K]}} |C_K(s)| = \frac{1}{|S|}\sum_{\substack{s\in S\\ r\in[s,K]}} \frac{1}{|[s,K]|}.$$

Further they found the following bounds for $\Pr_r(S,K)$.

Theorem 3.2. *[7, Proposition 3.1] Let S and K be two additive subgroups of R. If $r \neq 0$ then*

1. $\Pr_r(S,K) \geq \frac{|Z(S,K)||Z(K,S)|}{|S||K|}$.
2. *If $S \subseteq K$ then* $\Pr_r(S,K) \geq \frac{2|Z(S,K)||Z(K,S)|}{|S||K|}$.

Theorem 3.3. *[7, Proposition 3.4] If $S_1 \subseteq S_2$ and $K_1 \subseteq K_2$ are additive subgroups of R then*

$$\Pr_r(S_1, K_1) \leq |S_2 : S_1||K_2 : K_1|\Pr_r(S_2, K_2).$$

Theorem 3.4. *[7, Proposition 3.5] Let S, K_1 and K_2 be three additive subgroups of R. If $K_1 \subseteq K_2$ then*

$$\Pr(S, K_1) \geq \Pr(S, K_2) \geq \frac{1}{|K_2 : K_1|}\left(\Pr(S, K_1) + \frac{|K_2| - |K_1|}{|S||K_1|}\right).$$

The first equality holds if and only if $[s, K_1] = [s, K_2]$ for all $s \in S$ and the second equality holds if and only if $C_S(k) = \{0\}$ for all $k \in K_2 \setminus K_1$.

Theorem 3.5. *[7, Proposition 3.6] Let $S \subseteq K$ be two additive subgroups of R. If p is the smallest prime dividing $|R|$ and $|S : Z(S, K)| = p^n$ then $\Pr(S, K) \leq \frac{p^n+p-1}{p^{n+1}}$. Moreover, if $S = K$ then we have $\Pr(S, K) \geq \frac{p^n+p^{n-1}-1}{p^{2n-1}}$.*

Theorem 3.6. *[7, Theorem 3.1] Let S and K be two additive subgroups of R and p the smallest prime dividing $|R|$. Then*

$$\Pr(S, K) \geq \frac{|Z(S, K)|}{|S|} + \frac{p(|S| - |X_S| - |Z(S, K)|) + |X_S|}{|S||K|}$$
$$\text{and } \Pr(S, K) \leq \frac{(p-1)|Z(S, K)| + |S|}{p|S|} - \frac{|X_S|(|K| - p)}{p|S||K|}$$

where $X_S = \{s \in S : C_K(s) = \{0\}\}$. Moreover, in each of these bounds, S and K can be interchanged.

Theorem 3.7. *[7, Proposition 3.7] Let S and K be two additive subgroups of R. Then*

$$\Pr(S, K) \geq \frac{1}{|[S, K]|}\left(1 + \frac{|[S, K]| - 1}{|S : Z(S, K)|}\right).$$

In particular, if $Z(S, K) \neq S$ then $\Pr(S, K) > \frac{1}{|[S,K]|}$.

Dutta and Nath also characterize ring R in terms of the ratio $\Pr(S, K)$. The following results give some characterizations.

Theorem 3.8. *[7, Theorem 3.2] Let S and K be two additive subgroups of R such that $\Pr(S, K) = \frac{2p-1}{p^2}$ for some prime p. Then p divides $|R|$. If p is the smallest prime dividing $|R|$ then*

$$\frac{S}{Z(S, K)} \cong \mathbb{Z}_p \cong \frac{K}{Z(K, S)}$$

and hence $S \neq K$. In particular, if $\Pr(S, K) = \frac{3}{4}$ then

$$\frac{S}{Z(S, K)} \cong \mathbb{Z}_2 \cong \frac{K}{Z(K, S)}.$$

Theorem 3.9. *[7, Theorem 3.3] Let $S \subseteq K$ be two non-commutative additive subgroups of R and $\Pr(S,K) = \frac{p^2+p-1}{p^3}$ for some prime p. Then p divides $|R|$. If p is the smallest prime dividing $|R|$ then*

$$\frac{S}{Z(S,K)} \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

In particular, if $\Pr(S,K) = \frac{5}{8}$ then $\frac{S}{Z(S,K)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

The following theorem gives partial converse of Theorem 3.8 and Theorem 3.9.

Theorem 3.10. *[7, Theorem 3.4] Let $S \subseteq K$ be two additive subgroups of R.*

1. *If $\frac{S}{Z(S,K)} \cong \mathbb{Z}_p$ and $|K:S| = n$ then $\Pr(S,K) \geq \frac{n+p-1}{np}$. Further, if p is the smallest prime dividing $|R|$ and $|K:S| = p$ then $\Pr(S,K) = \frac{2p-1}{p^2}$.*

2. *If $\frac{S}{Z(S,K)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and $|K:S| = n$ then $\Pr(S,K) \geq \frac{(n+2)p^2-2}{np^4}$. Further, if p is the smallest prime dividing $|R|$ and $|K:S| = 1$ then $\Pr(S,K) = \frac{p^2+p-1}{p^3}$.*

Theorem 3.11. *[7, Theorem 3.5] Let S and K be two additive subgroups of R and I be an ideal of R such that $I \subseteq S \cap K$. Then*

$$\Pr(S,K) \leq \Pr\left(\frac{S}{I}, \frac{K}{I}\right) \Pr(I).$$

The equality holds if $I \cap [S,R] = \{0\}$.

We conclude with the following generalization of Theorem 2.7.

Theorem 3.12. *[7, Theorem 4.1] Let R_1 and R_2 be two finite non-commutative rings with additive subgroups S_1, K_1 and S_2, K_2 respectively. If $\phi_1 : \frac{S_1}{Z(S_1,R_1)} \to \frac{S_2}{Z(S_2,R_2)}$, $\phi_2 : \frac{K_1}{Z(K_1,R_1)} \to \frac{K_2}{Z(K_2,R_2)}$ and $\psi : [S_1,K_1] \to [S_2,K_2]$ are additive group isomorphisms such that*

$$a_{(S_2,K_2)} \circ (\phi_1 \times \phi_2) = \psi \circ a_{(S_1,K_1)},$$

where $a_{(S_i,K_i)} : \frac{S_i}{Z(S_i,R_i)} \times \frac{K_i}{Z(K_i,R_i)} \to [S_i,K_i]$ are well defined maps given by

$$a_{(S_i,K_i)}(x_i + Z(S_i,R_i), y_i + Z(K_i,R_i)) = [x_i, y_i]$$

for all $x_i \in S_i, y_i \in K_i$ and $i = 1,2$; and

$$(\phi_1 \times \phi_2)(x_1 + Z(S_1,R_1), y_1 + Z(K_1,R_1)) = (x_2 + Z(S_2,R_2), y_2 + Z(K_2,R_2))$$

whenever $\phi_1(x_1 + Z(S_1,R_1)) = x_2 + Z(S_2,R_2)$ and $\phi_2(y_1 + Z(K_1,R_1)) = y_2 + Z(K_2,R_2)$, then

$$\Pr_r(S_1,K_1) = \Pr_{\psi(r)}(S_2,K_2).$$

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A survey on graphs related to rings

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Abstract. In this chapter, we shall highlight the outcomes of a survey on graphs related to rings. Here we have studied graphs, which are found from a ring with a specific condition. Results stated here can be used to develop some further ideals. Here we mainly focus on graph-theoretic properties like girth, clique number, diameter, dominating set, chromatic index etc. of some graphs related to rings.

2010 Mathematical Sciences Classification. 16N40, 16U99.

Keywords. Zero divisor graph, Idempotent divisor graph, Nilpotent divisor graph, Nil clean graph of a ring, Nil clean ring, Weakly nil clean ring.

1 INTRODUCTION

Throughout our discussion, unless or otherwise explicitly stated, R will denote an associative ring with unity. We will use the symbols $Reg(R), Nil(R), U(R)$ and $Idem(R)$ respectively to denote the set of all von Neumann regular elements, nilpotent elements, units and idempotents of R. Also $J(R)$ will denote the Jacobson radical of R. Let M be a left R module. We denote the endomorphism ring of M by $end(M)$ and denote the ring of $n \times n$ matrices over the ring R by $M_n(R)$. A ring R is called a reduced ring if it has no non-zero nilpotent elements.

In ring theory, many mathematician have studied graph theoretic properties of a graph associated with a ring. Beck [5] studied colouring of a graph of a finite commutative ring R in 1988, where the vertex set is R and any two vertices x and y are adjacent if and only if $xy = 0$. Grimaldi [8] defined and studied a notion of graph $G(\mathbb{Z}_n)$, known as unit graph associated with $\mathbb{Z}_n$, the ring of integer modulo n. In this graph, the vertex set is $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x + y$ is unit. Further Ashrafi, Pournaki, Maimani and Yassemi [2] generalised $G(\mathbb{Z}_n)$ to $G(R)$, for any arbitrary associative ring R with non zero identity. An element r of a ring R is said to be a nil clean element of R, if r can be expressed as $r = e + n$, where $e \in Idem(R)$ and $n \in Nil(R)$, also R is said to be nil clean ring [7] if all the elements are nil clean element. The set of all nil clean elements of a ring R is denoted ny $NC(R)$. A ring R is said to be weakly nil clean ring [4, 6] if for any $x \in R$, $x = e + n$ or $x = e - n$, where $e \in Idem(R)$ and $n \in Nil(R)$. A ring R is called null ring if $R^2 = \{0\}$. In 2018, Basnet and Bhattacharyya [3] introduced nil clean graph

$G_N(R)$ associated with a finite commutative ring R, where the vertex set is R and two distinct vertices x and y are adjacent if and only if $x + y$ is nil clean element of R. In 2010, Li et al. [10] studied a kind of graph structure of a ring R, known as nilpotent divisor graph of R, whose vertex set is $\{x \in R : x \neq 0, \exists\, y(\neq 0) \in R$ such that xy is nilpotent$\}$ and two vertices x and y are adjacent if xy is nilpotent. In 2018, Kimball and LaGrange [9] generalized the concept of zero divisor graph to idempotent divisor graph. For any idempotent $e \in R$, they defined the idempotent divisor graph $\Gamma_e(R)$ associated with e, where $V(\Gamma_e(R)) = \{a \in R$: there exists $b \in R$ with $ab = e\}$ and two vertices a and b are adjacent if $ab = e$. We refer [1, 2, 5, 11] for more work on graph associated with rings.

2 A survey on graph related to rings

In this section, we mention some results related to graphs obtained from rings and its graph theoretic properties like girth, clique number, diameter, dominating set, chromatic index etc.

Preliminaries

Here we mention some preliminaries about graph theory. Let G be a graph. The degree of the vertex $v \in G$ denoted by $deg(v)$, is the number of edges adjacent with v. A graph G is said to be *connected* if for any two distinct vertices of G, there is a path in G connecting them. Number of edges on the shortest path between vertices x and y is called the *distance* between x and y and is denoted by $d(x, y)$. If there is no path between x and y then we say $d(x, y) = \infty$. The diameter of a graph G, denoted by $diam(G)$, is the maximum of distances of each pair of distinct vertices in G. Also *girth* of G is the length of the shortest cycle in G, denoted by $gr(G)$. Note that if there is no cycle in G then $gr(G) = \infty$. A complete graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A bipartite graph G is a graph whose vertices can be divided into two disjoint parts V_1 and V_2, such that $V(G) = V_1 \cup V_2$ and every edge in G has the form $e = (x, y) \in E(G)$, where $x \in V_1$ and $y \in V_2$. Note that no two vertices both in V_1 or both in V_2 are adjacent. A complete bipartite graph is a graph where every vertex of the first part V_1 is connected to every vertex of the second part V_2, denoted by $K_{m,n}$, where $|V_1| = m$ and $|V_2| = n$. A complete bipartite graph $K_{1,n}$ is called star graph.

A clique is a subset of vertices of an undirected graph such that its induced subgraph is complete. A clique having n number of vertices is called n-clique. The maximum clique of a graph is a clique such that there is no clique with more vertices. The clique number of a graph G is denoted by $\omega(G)$ and defined by the number of vertices in the maximal clique of G. A coclique in a graph G is a set of pairwise nonadjacent vertices. An edge colouring of a graph G is a map $C : E(G) \to S$, where S is a set of colours such that for all $e_1, e_2 \in E(G)$, if e_1 and e_2 are adjacent then $C(e_1) \neq C(e_2)$. The *chromatic index* of a graph G is denoted by $\chi'(G)$ and is defined as the minimum number of colours needed for a proper colouring of G. A dominating set for a graph G is a subset D of a vertex set of G such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G.

Some results on graphs related to rings

In this subsection, we give some results about graphs related to rings. These results will help for the future study about graphs related to rings, which will also be useful to extend the ideas of the

following graphs obtained from a ring.

- In 1986, I. Beck [5] present the idea of coloring of a commutative ring. This idea establishes a connection between graph theory and commutative ring theory which hopefully will turn out to be mutually beneficial for these two branches of mathematics. In this paper, Beck mainly interested in characterizing and discussing the rings which are finitely colorable, which will be a possible applications to graph theory. For a commutative ring R, he defined a graph whose vertex set is R and two vertices x and y are adjacent if $xy = 0$. This graph is known as a zero divisor graph of a commutative ring R. Some of his results are listed below:

Proposition 2.1. *(1) $\chi(R) = 1$ if and only if R is the zero ring.*

(2) $\chi(R) = 1$ if and only if R is an integral domain, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$.

Proposition 2.2. *Let $p_1, p_2, \cdots p_k, q_1, q_2, \cdots q_r$ are distinct prime numbers and $N = p_1^{2n_1}.p_2^{2n_2}.\cdots.p_k^{2n_k}.q_1^{2m_1+1}.q_2^{2m_2+1}.\cdots.q_r^{2m_r+1}$. Then*

$$\chi(\mathbb{Z}_N) = \omega(\mathbb{Z}_N) = p_1^{n_1}.p_2^{n_2}.\cdots.p_k^{n_k}.q_1^{m_1}.q_2^{m_2}.\cdots.q_r^{2m_r}$$

An element $x \in R$ is called finite if the ideal Rx is a finite set. Beck also studied rings having finite chromatic index.

Lemma 2.3. *(1) Suppose that R has an infinite number of finite elements. Then R contains an infinite clique.*

(2) Let I be a finite ideal of R. Then the ring R contains an infinite clique if and only if R/I has an infinite clique.

(3) If the ring contains a nilpotent element which is not finite then R contains an infinite clique.

(4) If the nil radical of R is infinite then R has an infinite clique.

Theorem 2.4. *For a reduced ring R, the following are equivalent.*

(1) $\chi(R)$ is finite.

(2) $\omega(R)$ is finite.

(3) The zero-ideal in R is the finite intersection of prime ideals.

(4) R does not contain an infinite clique.

He also proved the following main results.

Theorem 2.5. *Let R be a reduced ring$\neq (0)$. If $\chi(R) < \infty$, then R has only a finite number of minimal prime ideals. If n is this number then $\chi(R) = \omega(R) = n + 1$.*

Theorem 2.6. *Let R be a ring which contains a finite ideal which is a intersection of prime ideals. Then the radical of any finite ideal is finite and equals a finite intersection of prime ideals. Furthermore, the ring has only a finite number of finite ideals.*

Definition 2.7. *A ring R is called a Coloring provided $\chi(R)$ is finite.*

Given an ideal K, $rad\, K = \{e \in R \,|\, r^n \in K\}$. The set of zero divisors in R is denoted by $Z(R)$. A prime ideal P is an associated prime ideal if $P = Ann(x)$, for some $x \in R$. Here $Ass(R)$ denotes the set of all associated prime ideals of R.

Theorem 2.8. *Let R be a Coloring. Then $Ass(R)$ is finite and $Z(R) = \bigcup_{P \in Ass(R)} P$. Furthermore, any minimal prime ideal P is an associated prime ideal and R_P is a field or a finite ring.*

Theorem 2.9. *Let P be an associated prime ideal in a Coloring. Then either R_P is a field or P is a maximal ideal.*

Theorem 2.10. *1. A subring of a Coloring is also a Coloring.*

2. A finite product of Coloring is a Coloring.

Theorem 2.11. *1. Let I be a finite ideal in a Coloring R. Then R/I is a Coloring.*

2. Let I be a finitely generated ideal in a Coloring. Then $R/Ann(I)$ is a Coloring.

- In 1990, Grimaldi [8] defined and studied a notion of graph $G(\mathbb{Z}_n)$, known as unit graph associated with $\mathbb{Z}_n$, the ring of integer modulo n. In this graph, the vertex set is $\mathbb{Z}_n$ and two distinct vertices x and y are adjacent if and only if $x + y$ is unit.
- Further Ashrafi, Pournaki, Maimani and Yassemi [2] generalized the notion of unit graph $G(R)$ to any arbitrary ring R. For any ring R, the vertex set of $G(R)$ is R and two distinct elements x and y are adjacent if and only if their sum is a unit. Some of their results are stated below:

Proposition 2.12. *Let R be a finite ring. Then the following statements hold for the unit graph of R.*

(1) If $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph.

(2) If $2 \in U(R)$, then for every $x \in R \setminus U(R)$ we have $deg(x) = |U(R)|$.

Lemma 2.13. *Let R be a commutative ring. If $x, y \in R$, then the following statement hold:*

(1) If $x + J(R)$ and $y + J(R)$ are adjacent in $G(R/J(R))$, then every element of $x + J(R)$ is adjacent to every element of $y + J(R)$ in $G(R)$.

(2) If $2x \notin U(R)$, then $x + J(R)$ is a coclique in $G(R)$.

(3) If $2x \in U(R)$, then $x + J(R)$ is a clique in $G(R)$.

They also characterized unit graphs of rings. Some of the results about the characterization is given below:

Theorem 2.14. *Let R be a ring. Then the unit graph $G(R)$ is a complete graph if and only if R is a division ring with $Char(R) = 2$.*

Theorem 2.15. *Let R be a commutative ring and M be a maximal ideal of R such that $|R/M| = 2$. Then $G(R)$ is a bipartite graph. Moreover the unit graph $G(R)$ is a complete bipartite graph if and only if R is a local ring.*

Theorem 2.16. *Let R be a finite commutative ring. Then the unit graph $G(R)$ is planar if and only if R is isomorphic to one of the following rings.*

(1) $\mathbb{Z}_3$,

(2) $\mathbb{Z}_3 \times \mathbb{Z}_3$,

(3) $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2 \times S$ *(product occurs $l+1$-times and $l \geq 0$), where $S \cong \mathbb{Z}_2$ or $S \cong \mathbb{Z}_3$ or $S \cong \mathbb{Z}_4$ or $S \cong \mathbb{F}_4$ or $S \cong \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_2\}$.*

They also studied diameter and girth of unit graph of a ring.

Theorem 2.17. *If R is a finite commutative ring. Then $diam(G(R)) \in \{1, 2, 3, \infty\}$.*

Theorem 2.18. *Let,R be a finite commutative ring. Then the following statement hold:*

(1) $diam(G(R)) = 1$ if and only if R is a field with $Cgar(R) = 2$.

(2) $diam(G(R)) = 2$ if and only if one of the following cases occur:

(a) R is a field with $Char(R) \neq 2$.

(b) R is not a field and R cannot have $\mathbb{Z}_2$ as a quotient.

(c) R is local ring with maximal ideal M such that $|R/M| = 2$ and $R \ncong \mathbb{Z}_2$.

(3) $diam(G(R)) = 3$ if and only if R has quotient and cannot have $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient and R is not a local ring.

(4) $diam(G(R)) = \infty$ if and only if R has $\mathbb{Z}_2 \times \mathbb{Z}_2$ as a quotient.

Theorem 2.19. *Let R be a finite commutative ring. Then $gr(R) \in \{3, 4, 6, \infty\}$.*

- In 2017, D. Basnet and J. Bhattacharyya defined and studied the notion of nil clean graph of a finite commutative ring. For a finite commutative ring R, the nil clean graph of R is denoted by $G_N(R)$, where the vertex set is R and two vertices x and y are adjacent if and only if $x + y$ can be expressed as a sum of nil clean element and idempotent element of R. They have studied graph theoretic properties like girth, dominating set, diameter etc. for a nil clean graph of a finite commutative ring. Some of their results are the following:

Theorem 2.20. *The following hold for nil clean graph $G_N(R)$ of R:*

(1) If R is not a field, then the girth of $G_N(R)$ is equal to 3.

(2) girth is infinite otherwise.

Theorem 2.21. *Let R be a weakly nil clean ring then $\{1, 2\}$ is a dominating set for $G_N(R)$.*

Theorem 2.22. *If R be a finite commutative ring, then the nil clean graph of R is of class 1.*

Theorem 2.23. *Let R be a non clean ring, weakly nil clean ring with no non trivial idempotents then $diam(G_N(R)) = 2$.*

Theorem 2.24. *Let $R = A \times B$, such that A is nil clean but B is weakly nil clean with no non trivial idempotents, then $diam(G_N(R)) = 2$.*

Theorem 2.25. *Let n be a positive integer, then the following hold for $\mathbb{Z}_n$:*

(1) If $n = 2^k$, for some integer $k \geq 1$, then $diam(G_N(\mathbb{Z}_n)) = 1$.
(2) If $n = 2^k 3^l$, for some integer $k \geq 0$ and $l \geq 1$, then $diam(G_N(\mathbb{Z}_n)) = 1$.
(3) For a prime p, $diam(G_N(\mathbb{Z}_p)) = 1$
(4) If $n = 2p$, where p is an odd prime, then $diam(G_N(\mathbb{Z}_n)) = p - 1$.
(5) If $n = 3p$, where p is an odd prime, then $diam(G_N(\mathbb{Z}_n)) = p - 1$

- In 2010, Li et al. [10] studied a kind of graph structure $\Gamma_N(R)$ of a ring R, known as nilpotent divisor graph of R, whose vertex set is $\{x \in R : x \neq 0, \exists\, y(\neq 0) \in R$ such that xy is nilpotent$\}$ and two vertices x and y are adjacent if and only if xy is nilpotent. This graph is a natural generalization of zero divisor graph. Some of their results are given below:

Theorem 2.26. *Assume that R is a nonreduced commutative ring and $\Gamma_N(R)$ is not a singleton. Then the following statements are equivalent:*

(1) $gr(\Gamma_N(R)) = \infty$.
(2) $\Gamma_N(R)$ is a star graph.
(3) R is either a null ring of order 3, or $Nil(R)$ is a prime ideal of R with $|Nil(R)| = 2$.

Theorem 2.27. *Let R be a regular ring with identity 1. Then the following hold:*

(1) $\Gamma_N(R)$ is connected.
(2) $diam(\Gamma_N(R)) \leq 3$.
(3) If $\Gamma_N(R)$ contains a cycle, then $gr(\Gamma_N(R)) \leq 4$. Moreover if R is nonreduced, then $gr(\Gamma_N(R)) = 3$.

Theorem 2.28. *Assume that R is a nonreduced regular ring and $\Gamma_N(R)$ is a star graph. Then the following hold:*

(1) $\Gamma_N(R)$ has exactly two vertices if and only if R is a nilpotent ring with order 3.
(2) If $\Gamma_N(R)$ has at least three vertices, then $Nil(R)$ is a prime ideal of R with $|Nil(R)| = 2$.

Theorem 2.29. *Let R be a finite commutative ring. Then $\chi'(\Gamma_N(R)) = \Delta(\Gamma_N(R))$ unless R is a nilpotent ring with even order.*

Theorem 2.30. *Let R be a finite reduced commutative ring and S be a commutative ring which is not an integral domain. If $\Gamma_N(R) \cong \Gamma_N(S)$, then $R \cong S$, unless $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and S is a null ring of order 3, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ and S is isomorphic to either $R \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

Theorem 2.31. *Assume that R and S are finite nonreduced commutative rings such that $\Gamma_N(R) \cong \Gamma_N(S)$. Then $|R| = |S|$ and $|Nil(R)| = |Nil(S)|$.*

- In 2018, Kimball and LaGrange [9] generalized the concept of zero divisor graph to idempotent divisor graph. For any idempotent $e \in R$, they defined the idempotent divisor graph $\Gamma_e(R)$ associated with e, where $V(\Gamma_e(R)) = \{a \in R$: there exists $b \in R$ with $ab = e\}$ and two vertices a and b are adjacent if $ab = e$. Notice that if $e = 0$ then $\Gamma_e(R)$ is a idempotent divisor graph of R.

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A study on coprime graph of finite groups

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Abstract. Let G be a finite group and $a \in G$. We write $o(a)$ to denote the order of a, that is the smallest positive integer m such that a^m is the identity element of G. The coprime graph of G, denoted by $\mathcal{C}_G$, is a simple undirected graph whose vertex set is G and two distinct vertices x and y are adjacent if and only if $o(x)$ and $o(y)$ are relatively prime. In this chapter, we have studied some of the properties of coprime graphs $\mathcal{C}_G$ of finite groups. In particular, we have chosen dihedral group $D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$, and studied the properties of its coprime graph denoted by $\mathcal{C}_{D_{2n}}$ along with it we have also tried to determine characteristics polynomials of adjacency matrices, laplacian matrices and signless laplacian matrices of these coprime graphs, $\mathcal{C}_{D_{2n}}$.

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1 Introduction

The coprime graph of a finite group G is a simple undirected graph whose vertex set is G and two distinct vertices x and y are adjacent if and only if $o(x)$ and $o(y)$ are relatively prime, we shall denote it by $\mathcal{C}_G$.

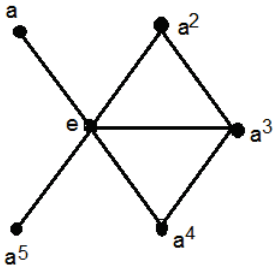

Fig. :1

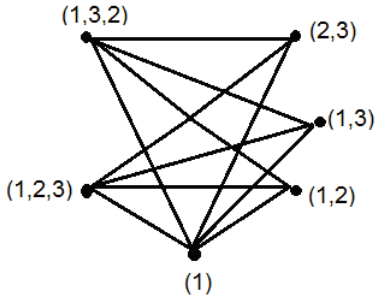

Fig. :2

For example, Fig.1 and Fig.2 are the coprime graphs of Z_6 and S_3 respectively.

The coprime graph of a group G was introduced and studied by Ma, Wei and Yang [2] in the year 2014, later this graph was further studied by Dorbidi [1] in 2016. They have obtained the following results.

Proposition 1.1. [2, Proposition 2.1] *Let G be any group. Then $diam(\mathcal{C}_G) \leq 2$. In particularly, $\mathcal{C}_G$ is connected and the girth of $\mathcal{C}_G$ equals 3 or ∞.*

Proposition 1.2. [2, Proposition 2.3] *Let G be a group. Then $diam(\mathcal{C}_G) = 1$ if and only if G is isomorphic to cyclic group Z_2 with order 2.*

Corollary 1.3. [2, Corollary 2.4] *The $\mathcal{C}_G$ is regular if and only if the group G is isomorphic to Z_2.*

Corollary 1.4. [2, Corollary 2.5] *The $\mathcal{C}_G$ is not complete whenever the group G is of order greater than 2.*

Theorem 1.5. [1, Theorem 3.2] *The $\mathcal{C}_G$ is a complete r-partite graph if and only if the order of every non-identity element of the group G is a prime power and $r = |\pi(G)| + 1$, where $\pi(G)$ is the set of prime divisors of $o(G)$.*

Proposition 1.6. [2, Proposition 2.11] *Let G be a cyclic group with order $2p$ for some odd prime p. Then $\mathcal{C}_G$ is planar.*

Theorem 1.7. [1, Theorem 3.6] *The $\mathcal{C}_G$ is a planar graph if and only if the group G is a p-group or $G \cong Z_2 \times Q$, where Q is a q-group.*

Proposition 1.8. [2, Proposition 2.13] *If two groups G_1 and G_2 are isomorphic, that is, $G_1 \cong G_2$, then their coprime graphs are also isomorphic, that is, $\mathcal{C}_{G_1} \cong \mathcal{C}_{G_2}$.*

Remark 1.9. [2, Remark 2.14] *The converse of [Proposition 1.8] is not true. Let $G_1 = D_8$ and let $G_2 = Z_8$. We see that G_1 and G_2 are 2-groups. Clearly $\mathcal{C}_{G_1} \cong \mathcal{C}_{G_2}$, but $G_1 \ncong G_2$.*

Theorem 1.10. [2, Theorem 3.1] *Let G be a group of order $p_1^{r_1}p_2^{r_2}...p_n^{r_n}$, where p_i is a prime for every $i \in \{1, 2, ..., n\}$ and r_i is a non-negative integer for every $i \in \{1, 2, ..., n\}$. Then $\mathcal{C}_G$ has no end-vertex if and only if G has no elements of order $p_1^{k_1}p_2^{k_2}...p_n^{k_n}$, where $1 \leq k_i \leq r_i$.*

Proposition 1.11. [2, Proposition 3.3] *Let G be a group of order n, where $n \geq 3$. If G is cyclic, then $\mathcal{C}_G$ contains some end-vertex. Particularly, the number of end-vertices of $\mathcal{C}_G$ is greater than or equal to $\phi(n)$.*

Remark 1.12. [2, Remark 3.4] *In general, the converse of [Proposition 1.11] is false. Such as the Klein 4-group $K_2 \times K_2$ or the dihedral group D_8, they are non-cyclic. However, $\mathcal{C}_{K_2\times K_2}$ and $\mathcal{C}_{D_8}$ have 3 end-vertices and 7 end-vertices, respectively. More specifically that every p-group of non-cyclic is a counter-example.*

The following theorem is a generalization of [Proposition 1.11].

Theorem 1.13. [1, Theorem 3.8] *If G is a nilpotent group of order n then $\mathcal{C}_G$ has $f(n)$ end vertices, where $f(n) = \sum_{d|n} \phi(d)$.*

Theorem 1.14. [2, Theorem 3.5] *Let G be a group with order greater than 2. Then $\mathcal{C}_G$ contains precisely two end-vertices if and only if G is isomorphic to Z_3 or Z_6, or a non-cyclic group G satisfying the following conditions:*

1. *$\pi(G) = \{2, 3\}$, where $\pi(G)$ is the set of prime divisors of $o(G)$;*
2. *G contains two elements x and y, such that $o(x) = o(y) = 6$ and $y = x^{-1}$;*
3. *$o(g) < 6$ for every $g \in G$, where $g \neq x, y$.*

2 Coprime graph of dihedral groups

For $n \geq 3$, the dihedral group $D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1}\rangle$. That is, $D_{2n} = \{a^1, a^2, a^3, ..., a^n = e, a^1b, a^2b, a^3b, ..., a^nb\}$. Fig.3 and Fig.4 are the coprime graphs of D_6 and D_8 respectively.

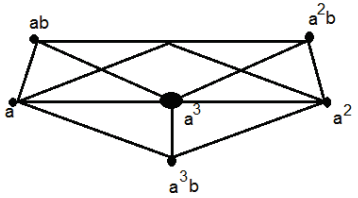

Fig. :3

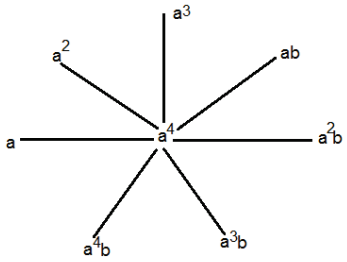

Fig. :4

Below we have mentioned some theoretical properties which hold for the coprime graphs of the dihedral groups, such as degree, traversability, planarity etc. derived by Ma, Wei and Yang [2].

Theorem 2.1. [2, Theorem 5.1] *Let $\mathcal{C}_{D_{2n}}$ be the coprime graph of D_{2n} and let n be odd. Then*

1. *$deg(a^ib) = n$ for any $1 \leq i \leq n$;*
2. *$deg(a^i) \geq n$ for any $1 \leq i \leq n$;*
3. *$\mathcal{C}_{D_{2n}}$ is not Eulerian;*
4. *$\mathcal{C}_{D_{2n}}$ is Hamiltonian;*
5. *$\mathcal{C}_{D_{2n}}$ is not planar.*

Corollary 2.2. [2, Corollary 5.2] *Let n be an odd prime. Then $\mathcal{C}_{D_{2n}} \cong K_{1,n-1,n}$.*

Theorem 2.3. [2, Theorem 5.3] *Let, $n = 2^k p_1^{r_1} p_2^{r_2} ... p_m^{r_m}$, where p_i is a prime integer and r_i is a non-negative integer for any $1 \leq i \leq m$ and k is a positine integer. Then*

1. *The number of end-vertices of $\mathcal{C}_{D_{2n}}$ is $\sum_{d|n} \phi(d)$, where $2p_1p_2...p_m$ is a divisor of d. In particular, $\mathcal{C}_{D_{2n}}$ contains an end-vertices;*
2. *$\mathcal{C}_{D_{2n}}$ is not Eulerian;*
3. *$\mathcal{C}_{D_{2n}}$ is not Hamiltonian;*
4. *$\mathcal{C}_{D_{2n}}$ is not planar.*

Corollary 2.4. [2, Corollary 5.4] *Let $n = 2^k$ for some positive integer k. Then $\mathcal{C}_{D_{2n}} \cong K_{1,2^{k+1}-1}$.*

Corollary 2.5. [2, Corollary 5.5] *$\mathcal{C}_{D_{2n}}$ is planar if and only if $n = 2^k$ for some positive integer k.*

3 Computing characteristics polynomial of $A(\mathcal{C}_{D_{2n}})$

In this section we compute characteristics polynomial and spectrum(if possible) of adjacency matrices of $\mathcal{C}_{D_{2n}}$ denoted by charpoly$(A(\mathcal{C}_{D_{2n}}))$ and Spec$(\mathcal{C}_{D_{2n}})$ respectively for different forms of n.

We consider the following three cases.

Case 1: n is odd

For $n = 3$, we have

$$A(\mathcal{C}_{D_6}) = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Then charpoly$(A(\mathcal{C}_{D_6})) = \lambda^3(\lambda^3 - 11\lambda - 12)$.

For $n = 5$, we have

$$A(\mathcal{C}_{D_{10}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\text{charpoly}(A(\mathcal{C}_{D_{10}})) = \lambda^7(\lambda^3 - 29\lambda - 40)$.

For $n = 7$, we have

$$A(\mathcal{C}_{D_{14}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\text{charpoly}(A(\mathcal{C}_{D_{14}})) = \lambda^{11}(\lambda^3 - 55\lambda - 84)$.

For $n = 9$, we have

$$A(\mathcal{C}_{D_{18}}) = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\ 1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0 \end{bmatrix}.$$

Then $\text{charpoly}(A(\mathcal{C}_{D_{18}})) = \lambda^{15}(\lambda^3 - 89\lambda - 144)$.

Conjecture 3.1. *If n is odd then*

$$\text{charpoly}(A(\mathcal{C}_{D_{2n}})) = \lambda^{2n-3}(\lambda^3 - (n^2 + n - 1)\lambda - 2n(n-1)).$$

The cubic equations in the characteristics polynomials are a bit difficult to solve so spectrums are not determined at this moment.

Case 2: n is even

We consider the following two sub-cases.

Sub-case 2.1: $n = 2^k$ for some positive integer k

In this case, $\mathcal{C}_{D_{2^{k+1}}}$ becomes a star graph with 2^{k+1} vertices. Therefore,

$$\text{charpoly}(A(\mathcal{C}_{D_{2^{k+1}}})) = \lambda^{2^{k+1}-2}(\lambda^2 - 2^{k+1} + 1).$$

Hence

$$\text{Spec}(\mathcal{C}_{D_{2^{k+1}}}) = \{(\sqrt{2^{k+1} - 1})^1, 0^{2^{k+1}-2}, (-\sqrt{2^{k+1} - 1})^1\}.$$

Subcase 2.2: $n = 2^k p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$, where p_i is prime and r_i is a non-negative integer for every $i \in \{1, 2, ..., n\}$

For $n = 6$ we have

$$A(\mathcal{C}_{D_{12}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\text{charpoly}(A(\mathcal{C}_{D_{12}})) = \lambda^8(\lambda+2)(\lambda^3 - 2\lambda^2 - 21\lambda + 14)$.

For $n = 10$, we have

$$A(\mathcal{C}_{D_{20}}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\text{charpoly}(A(\mathcal{C}_{D_{20}})) = (\lambda - 1)^{16}(\lambda^4 - 63\lambda^2 - 88\lambda + 176)$.

For $n = 12$, we have

$$A(\mathcal{C}_{D_{24}}) = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&1&0&0&1&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

Then charpoly$(A(\mathcal{C}_{D_{24}})) = (\lambda-1)^{20}(\lambda^4 - 53\lambda^2 - 60\lambda + 180)$.

Since cubic and quartic equations appear in the characteristics polynomials which seem difficult to solve so spectrums cannot be determined at this stage. Due to dissimilar behaviour of the characteristics polynomials we cannot predict them in general.

4 Computing characteristics polynomial of $L(\mathcal{C}_{D_{2n}})$

In this section we compute characteristics polynomial and laplacian spectrum(if possible) of laplacian matrices of $\mathcal{C}_{D_{2n}}$ denoted by charpoly$(L(\mathcal{C}_{D_{2n}}))$ and $\Lambda - spec(\mathcal{C}_{D_{2n}})$ respectively for different forms of n.

We consider the following three cases.

Case 1: n is odd

For $n = 3$, we have

$$L(\mathcal{C}_{D_6}) = \begin{bmatrix} 4 & 0 & -1 & -1 & -1 & -1 \\ 0 & 4 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 \\ -1 & -1 & -1 & 3 & 0 & 0 \\ -1 & -1 & -1 & 0 & 3 & 0 \\ -1 & -1 & -1 & 0 & 0 & 3 \end{bmatrix}.$$

Then charpoly$(L(\mathcal{C}_{D_6})) = \lambda(\lambda-3)^2(\lambda-4)(\lambda-6)^2$. Therefore

$$\Lambda - spec(\mathcal{C}_{D_6}) = \{0, 3^2, 4^1, 6^2\}.$$

For $n = 5$, we have

$$L(\mathcal{C}_{D_{10}}) = \begin{bmatrix} 6 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 6 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 6 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 5 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 5 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 5 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 5 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then charpoly$(L(\mathcal{C}_{D_{10}})) = \lambda(\lambda-5)^4(\lambda-6)^3(\lambda-10)^2$. Therefore

$$\Lambda - spec(\mathcal{C}_{D_{10}}) = \{0, 5^4, 6^3, 10^2\}.$$

For $n = 7$, we have

$$L(\mathcal{C}_{D_{14}}) = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 8 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 8 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 8 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 8 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 8 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 13 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

Then charpoly$(L(\mathcal{C}_{D_{14}})) = \lambda(\lambda-7)^6(\lambda-8)^5(\lambda-14)^2$. Therefore

$$\Lambda - spec(\mathcal{C}_{D_{14}}) = \{0, 7^6, 8^5, 14^2\}.$$

For $n = 9$, we have

$$L(\mathcal{C}_{D_{18}}) = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 17 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}.$$

Then charpoly$(L(\mathcal{C}_{D_{18}})) = \lambda(\lambda-9)^8(\lambda-10)^7(\lambda-18)^2$. Therefore

$$\Lambda - spec(\mathcal{C}_{D_{18}}) = \{0, 9^8, 10^7, 18^2\}.$$

Conjecture 4.1. *If n is odd then*

$$\text{charpoly}(L(\mathcal{C}_{D_{2n}})) = \lambda(\lambda-n)^{n-1}(\lambda-(n+1))^{n-2}(\lambda-2n).$$

Therefore

$$\Lambda - spec(\mathcal{C}_{D_{2n}}) = \{0, n^{n-1}, (n+1)^{n-2}, 2n\}.$$

Case 2: n is even

We consider the following two subcases.

Subcase 2.1: $n = 2^k$ for some positive integer k

For $n = 4$, we have

$$L(\mathcal{C}_{D_8}) = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, charpoly$(L(\mathcal{C}_{D_8})) = \lambda(\lambda-1)^6(\lambda-8)$. Hence,

$$\Lambda - spec(\mathcal{C}_{D_8}) = \{0, 1^6, 8\}.$$

For $n = 8$, we have

$$L(\mathcal{C}_{D_{16}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 15 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, charpoly$(L(\mathcal{C}_{D_{16}})) = \lambda(\lambda-1)^{14}(\lambda-16)$. Hence,

$$\Lambda - spec(\mathcal{C}_{D_{16}}) = \{0, 1^{14}, 16\}.$$

For $n = 16$, we have charpoly$(L(\mathcal{C}_{D_{32}})) = \lambda(\lambda-1)^{30}(\lambda-32)$. Hence,

$$\Lambda - spec(\mathcal{C}_{D_{32}}) = \{0, 1^{30}, 32\}.$$

Conjecture 4.2. *If $n = 2^k$ then*

$$\text{charpoly}(L(\mathcal{C}_{D_{2^{k+1}}})) = \lambda(\lambda-1)^{2^{k+1}-2}(\lambda-2^{k+1}).$$

Therefore,

$$\Lambda - spec(\mathcal{C}_{D_{2^{k+1}}}) = \{0, 1^{2^{k+1}-2}, 2^{k+1}\}.$$

Subcase 2.2: $n = 2^k p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$, where p_i is prime and r_i is a non-negative integer for every $i \in \{1, 2, ..., n\}$

For $n = 6$ we have

$$L\mathcal{C}_{D_{12}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 8 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 11 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & -1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then charpoly$(L(\mathcal{C}_{D_{12}})) = \lambda(\lambda-1)^2(\lambda-3)^6(\lambda-8)(\lambda-10)(\lambda-12)$. Therefore

$$\Lambda - spec(\mathcal{C}_{D_{12}}) = \{0, 1^2, 3^6, 8, 10, 12\}.$$

For $n = 10$, we have

charpoly$(L(\mathcal{C}_{D_{20}})) = \lambda^2(\lambda-1)^2(\lambda-5)^{10}(\lambda-12)^2(\lambda-20)(\lambda^3-17\lambda^2+16\lambda+132)$.

It is a cubic equation which is bit difficult to solve at this stage so we are unable to compute the laplacian spectrums.

In this case due to dissimilar behaviour of the characteristics polynomials we cannot predict them in general.

5 Computing characteristics polynomial of $Q(\mathcal{C}_{D_{2n}})$

In this section we compute characteristics polynomial and signless laplacian spectrum(if possible) of signless laplacian matrices of $\mathcal{C}_{D_{2n}}$ denoted by charpoly$(Q(\mathcal{C}_{D_{2n}}))$ and Q-Spec$(\mathcal{C}_{D_{2n}})$ respectively for different forms of n.

We consider the following three cases.

Case 1: n is odd

For $n = 3$, we have

$$Q(\mathcal{C}_{D_6}) = \begin{bmatrix} 4 & 0 & 1 & 1 & 1 & 1 \\ 0 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 & 0 & 0 \\ 1 & 1 & 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 0 & 0 & 3 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_6})) = (\lambda-4)(\lambda-3)^2(\lambda^3-12\lambda^2+36\lambda-24)$.

For $n = 5$, we have

$$Q(\mathcal{C}_{D_{10}}) = \begin{bmatrix} 6 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 6 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 6 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 6 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 9 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 5 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 5 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 5 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{10}})) = (\lambda-6)^3(\lambda-5)^4(\lambda^3-20\lambda^2+100\lambda-80)$.

For $n = 7$, we have

$$Q(\mathcal{C}_{D_{14}}) = \begin{bmatrix} 8 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 8 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 8 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 8 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 8 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 13 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{14}})) = (\lambda-8)^5(\lambda-7)^6(\lambda^3-28\lambda^2+196\lambda-168)$.
For $n = 9$, we have

$$Q(\mathcal{C}_{D_{18}}) = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 10 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 17 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{18}})) = (\lambda-10)^7(\lambda-9)^8(\lambda^3-36\lambda^2+324\lambda-288)$.

Conjecture 5.1. *If n is odd then*

$$\text{charpoly}(Q(\mathcal{C}_{D_{2n}})) = (\lambda-n-1)^{n-2}(\lambda-n)^{n-1}(\lambda^3-4n\lambda^2+2n(n+1)\lambda-4n(n-1)).$$

As we can observed that the characteristics polynomials contain cubic equations which are very complicated to rationalise thus signless laplacian spectrums are not computed at this level.

Case 2: n is even

We consider the following two subcases.

Subcase 2.1: $n = 2^k$ for some positive integer k

For $n = 4$, we have

$$Q(\mathcal{C}_{D_8}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 7 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $\text{charpoly}(Q(\mathcal{C}_{D_8})) = \lambda(\lambda-1)^6(\lambda-8)$. Hence,

$$\text{Q-Spec}(\mathcal{C}_{D_8}) = \{0, 1^6, 8\}.$$

For $n = 8$, we have

$$Q(\mathcal{C}_{D_{16}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 15 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, $\text{charpoly}(Q(\mathcal{C}_{D_{16}})) = \lambda(\lambda-1)^{14}(\lambda-16)$. Hence,

$$\text{Q-Spec}(\mathcal{C}_{D_{16}}) = \{0, 1^{14}, 16\}.$$

For $n = 16$, we have $\text{charpoly}(Q(\mathcal{C}_{D_{32}})) = \lambda(\lambda-1)^{30}(\lambda-32)$. Hence,

$$\text{Q-Spec}(\mathcal{C}_{D_{32}}) = \{0, 1^{30}, 32\}.$$

Conjecture 5.2. *If $n = 2^k$ then*

$$\text{charpoly}(Q(\mathcal{C}_{D_{2^{k+1}}})) = \lambda(\lambda-1)^{2^{k+1}-2}(\lambda-2^{k+1}).$$

Therefore,

$$\text{Q-Spec}(\mathcal{C}_{D_{2^{k+1}}}) = \{0, 1^{2^{k+1}-2}, 2^{k+1}\}.$$

Subcase 2.2: $n = 2^k p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$, where p_i is prime and r_i is a non-negative integer for every $i \in \{1, 2, ..., n\}$

For $n = 6$ we have

$$Q(\mathcal{C}_{D_{12}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 8 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 11 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{12}})) = (\lambda - 1)^2(\lambda - 3)^6(\lambda - 8)(\lambda^3 - 22\lambda^2 + 120\lambda - 56)$.

For $n = 10$, we have

$$Q(\mathcal{C}_{D_{20}}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 12 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 12 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 12 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 19 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{20}})) = (\lambda-1)^2(\lambda-5)^{10}(\lambda-12)^2(\lambda^6-37\lambda^5+356\lambda^4-364\lambda^3-2222\lambda^2+2926\lambda-264)$.

For $n = 12$, we have

$$Q(\mathcal{C}_{D_{24}}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 16 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 16 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 23 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{bmatrix}.$$

Then charpoly$(Q(\mathcal{C}_{D_{24}})) = (\lambda-1)^6(\lambda-3)^{14}(\lambda-16)(\lambda^3-42\lambda^2+432\lambda-120)$.

For $n = 14$, we have
charpoly$(Q(\mathcal{C}_{D_{28}})) = (\lambda-1)^6(\lambda-7)^{14}(\lambda-16)^5(\lambda^3-50\lambda^2+616\lambda-360)$.

We have come across higher degree equations which are bit difficult to solve at this stage so we are unable to compute the signless laplacian spectrums and also due to dissimilar behaviour of the characteristics polynomials we cannot predict in general.

6 Conclusion

In this chapter we have studied the coprime graphs of finite groups and then we have particularly discussed the coprime graphs of dihedral groups. These classes of graphs are relatively new in the theory of finite groups. Therefore further research on the algebraic structures of these graphs can be carried out. Note that X. Ma, H. Wei and L. Yang have derived the concept of coprime graphs of finite groups and we have observed that very few results concerning the coprime graphs of dihedral groups are provided by them. Going through them we have tried to determine the characteristics polynomials and spectrums of the adjacency matrices, laplacian matrices, signless laplacian matrices of the coprime graphs of dihedral groups. To some extent we are successful in the cases when n is odd and $n = 2^k$ for some positive integer k. But whenever $n = 2^k p_1^{r_1} p_2^{r_2} ... p_n^{r_n}$,

where p_i is prime for every $1 \leq i \leq n$ and r_i is a non-negative integer for every $1 \leq i \leq n$ we have seen large variations in their characteristics polynomials so we posed there.

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7 Bibliography

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A Study of Genus of the Commuting Graphs

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Abstract. The commuting graph is a graph associtaed with the group. Commuting graph of a non-abelian group is defined as a simple graph in which the vertices are the non-central elements of the group, and two distinct vertices are adjacent if and only if they commute. In this article, it is discussed about (up to isomorphism) all finite non-abelian groups whose commuting graphs are acyclic, planar or toroidal. it is also discussed the explicit formulas for the genus of the commuting graphs of some well-known class of finite non-abelian groups, and it is shown that, every collection of isomorphism classes of finite non-abelian groups whose commuting graphs have the same genus is finite.

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Keywords. commuting graph, genus of the graph.

1 Introduction

This chapter is mainly focused on the graph which can be obtained from a non commuting group. There are so many graphs associated with the group, viz. commuting graph, non commuting graph, nilpotent graph, non nilpotent graph, prime graph, commuting conjugacy class graph, nilpotent conjugacy class graph. From the name of these graph we can predict that somehow these graphs are mostly related to the abelian group, nilpotent group, conjugacy classes of commuting group, conjugacy classes of nilpotent group etc. Here we associate the groups with the graph, for example; to define the Commuting or non-commuting graph, we take those groups which are not abelian and similarly for the other graphs also. To study the graph we basically study the property like connectedness, girth, independent number, domination number, genus. The reader may refer to [25] and [18] for various standard graph theoretic terminologies.

Let G be a non-abelian group and $Z(G)$ be its center. The commuting graph of G, denoted by $\Gamma_c(G)$, is a simple undirected graph in which the vertex set is $G \setminus Z(G)$, and two distinct vertices x and y are adjacent if and only if $xy = yx$. The reason to exclude the central element while defining the graph is very obvious, because the central element will give us a complete block, and give a

complete bipartite with the rest of the element. This graph is precisely the complement of the non-commuting graph of a group considered in [1] and [2]. However, the ever-increasing popularity of the topic is often attributed to a question, posed in 1975 by Paul Erdos and answered afirmatively by B. H. Neumann [19], asking whether or not a non-commuting graph having no infinite complete subgraph possesses a finite bound on the cardinality of its complete subgraphs. In recent years, the commuting graphs of groups have become a topic of research for many mathematicians (see, for example, [4],). In [5], it was conjectured that the commuting graph of a finite group is either disconnected or has diameter bounded above by a constant independent of the group G. This conjecture was well-supported in [6] and [7]. However, in [8], it is shown that, for all positive integers d, there exists a finite special 2-group G such that the commuting graph of G has diameter greater than d. But in [9], it is proved that for finite groups with trivial center the conjecture made in [5] holds good. The concept of commuting graphs of groups (taking, as the vertices, the non-trivial elements of the group in place of non-central elements) has also been recently used in [10] to show that finite quotients of the multiplicative group of a finite dimensional division algebra are solvable. There is also a ring theoretic version of commuting graphs (see, for example, [1], [11]).

Most of the works cited above on the commuting graphs of groups deal with connectedness, diameter and some algebraic aspects of the graph. Also, we deal with a topological aspect, namely, the genus of the commuting graphs of finite non-abelian groups, and on this concern the commuting and the non-commuting graphs are independent of each other. Here it is shown that every collection of isomorphism classes of finite non-abelian groups whose commuting graphs have the same genus is finite. The motivation for this survey comes from [12], [13], [14] and [15], where similar problems for certain graphs associated to finite rings have been addressed.

Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. Let $x, y \in V(\Gamma)$. Then x and y are said to be adjacent if $x \neq y$ and there is an edge $x \rightsquigarrow y$ in $E(\Gamma)$ joining x and y. A path between x and y is a sequence of adjacent vertices, often written as $x \rightsquigarrow x_1 \rightsquigarrow x_2 \rightsquigarrow \cdots \rightsquigarrow x_n \rightsquigarrow y$, where the vertices $x, x_1, x_2, \cdots x_n, y$ are all distinct (except, possibly, x and y). Γ is said to be connected if there is a path between every pair of distinct vertices in Γ. A path between x and y is called a cycle if $x = y$. The number of edges in a path or a cycle, is called its length. A cycle of length n is called an n-cycle, and a 3-cycle is also called a triangle. The girth of Γ is the minimum of the lengths of all cycles in Γ, and is denoted by $girth(\Gamma)$. If Γ is acyclic, that is, if Γ has no cycles, then we write $girth(\Gamma) = \infty$.

A graph G is said to be complete if there is an edge between every pair of distinct vertices in G. We denote the complete graph with n vertices by K_n. A bipartite graph is the one whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts. Among the bipartite graphs, the complete bipartite graph is the one in which two distinct vertices are adjacent if and only if they lie in different parts. The complete bipartite graph, with parts of size m and n, is denoted by $K_{m,n}$.

A subset of the vertex set of a graph Γ is called a clique of Γ if it consists entirely of pairwise adjacent vertices. The least upper bound of the sizes of all the cliques of G is called the clique number of Γ, and is denoted by $\omega(\Gamma)$. The chromatic number of a graph Γ, written $\chi(\Gamma)$, is the minimum number of colors needed to label the vertices so that adjacent vertices receive different colors. Clearly, $\omega(\Gamma) \leq \chi(\Gamma)$. Given a graph Γ, let U be a nonempty subset of $V(\Gamma)$. Then the induced subgraph of Γ on U is defined to be the graph $\Gamma[U]$ in which the vertex set is U and the edge set consists precisely of those edges in Γ whose endpoints lie in U.

We start with a conjecture given below.

Conjecture 1.1. *Let G and H be two non-abelian finite groups such that $\Gamma_c(G) \cong \Gamma_c(H)$. Then*

$G \cong H$.

2 Genus of $\gamma(\Gamma_c(G))$

The *genus* of a graph Γ, denoted by $\gamma(\Gamma)$, is the smallest non-negative integer g such that the graph can be embedded on the surface obtained by attaching g handles to a sphere. Clearly, if $\bar{\Gamma}$ is a subgraph of Γ then $\gamma(\bar{\Gamma}) \leq \gamma(\Gamma)$. Graphs having genus zero are called *planar* graphs, while those having genus one are called *toroidal* graphs. Graphs having genus two are called *double-toroidal* graphs and those having genus three are called *triple-toroidal* graph. It is well-known (see [24, Theorem 6-39]) that $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$, where $n \geq 3$ and K_n is the complete graph on n vertices. Also, if $m, n \geq 2$ then

$$\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil \text{ and } \gamma(K_{m,m,m}) = \frac{(m-2)(m-1)}{2},$$

where $K_{m,n}$, $K_{m,m,m}$ are complete bipartite and tripartite graphs respectively.

Lemma 2.1. *[21] If a graph Γ has two disjoint subgraphs Γ_1 and Γ_2 such that $\Gamma_1 = K_m$ and $\Gamma_2 = K_n$ for some positive integers m and n, then $\gamma(\Gamma) \geq \gamma(\Gamma_1) + \gamma(\Gamma_2)$.*

The above lemma is very usefull to find the bound of the genus, Since it is not possible always to find the genus by exactly knowing the graph, hence we can find the bound by using the above results.

Proposition 2.2. *[20] Let G be a non-abelian group. Then, $\Gamma_c(G)$ has no 3-cycle if and only if G is isomorphic to the symmetric group S_3, the quaternion group Q_8, or the dihedral group D_8.*

Lemma 2.3. *[20] Let G be a finite non-solvable group such that $\gamma(\Gamma_c(G)) = m$.*

1. *If S is a nonempty subset of $G \setminus Z(G)$ such that $xy = yx$ for all $x, y \in S$, then $|S| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$.*
2. *$|Z(G)| \leq \frac{1}{t-1} \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor$, where $t = \max\{o(xZ(G)) \mid xZ(G) \in G/Z(G)\}$.*
3. *If H is a abelian subgroup of G, then $|H| \leq \left\lfloor \frac{7+\sqrt{1+48m}}{2} \right\rfloor + |H \cap Z(G)|$.*

Theorem 2.4. *[20] The order of a finite non-abelian group is bounded by a function of the genus of its commuting graph. Consequently, given a non-negative integer g, there are at the most finitely many (up to isomorphism) finite non-abelian groups whose commuting graphs have genus g.*

A group is said to be an AC-group if the centralizer of each of its non-central elements is abelian. We have some result id the AC group. The clear example of this typs of groups are $D_{2n} = \langle x, y | y^n = x^2 = 1, xyx^{-1} = y^{-1} \rangle$, $Q_{4n} = \langle x, y | y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1} \rangle$, $\gamma(\Gamma_c(PSL(2, 2^k))) = (2^k + 1)\gamma(K_{2^k-1}) + 2^{k-1}(2^k + 1)\gamma(K_{2^k-2}) + 2^{k-1}(2^k - 1)\gamma(K_{2^k})$. The AC-groups have been extensively studied by many authors (see for example [1], [16]).

Proposition 2.5. *[20] Let G be a finite non- abelian $AC-$group. Then*

$$\gamma(\Gamma_c(G)) = \Sigma_{X\in\wp}\gamma(K_{|X|}),$$

where $\wp = \{C_G(u) \setminus Z(G) | u \in G \setminus Z(G)\}$.

Remark 2.6. *[20] If G is a finite non-abelian $AC-$ group and A is a finite abelian group, then $A \times G$ is also a finite non-abelian $AC-$ group with $C_{A\times G}(a,u) \setminus Z(A \times G) = A \times (C_G(u) \setminus Z(G))$ for all $(a,u) \in (A \times G) \setminus Z(A \times G)$. Therefore, it follows from previous Proposition that*

$$\gamma(\Gamma_c(A \times G)) = \Sigma_{X\in\wp}\gamma(K_{|A||X|}),$$

where $\wp = \{C_G(u) \setminus Z(G) | u \in G \setminus Z(G)\}$.

Corollary 2.7. *[20] The genus of the commuting graph of a non-abelian group G of order pq, where p and q are primes with $p|q-1$, is given by*

$$\gamma(\Gamma_c(G)) = \gamma(K_{q-1}) + q\gamma(K_{p-1}).$$

The genus of the commuting graphs of some well known finite non-abelian $AC-groups$ are detremined here. Some of the results obtained here play crucial role in the study of planarity and toroidality of the commuting graphs of finite nonabelian groups.

Corollary 2.8. *[20] The genus of the commuting graph of a non-abelian group G of order p^3, where p is a prime, is given by*

$$\gamma(\Gamma_c(G)) = (p+1)\gamma(K_{p(p-1)}).$$

Proposition 2.9. *[20] The genus of the commuting graph of the dihedral group $D_{2n} = \langle x, y | y^n = x^2 = 1, xyx^{-1} = y^{-1}\rangle$, where $n \geq 3$, is given by*

$$\gamma(\Gamma_c(D_{2n})) = \begin{cases} \gamma(K_{n-2}) & \text{if } n \text{ is even,} \\ \gamma(K_{n-1}) & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 2.10. *[20] The genus of the commuting graph of the dicyclic group or the generalized quaternion group $Q_{4n} = \langle x, y | y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, where $n \geq 2$, is given by*

$$\gamma(\Gamma_c(Q_{4n})) = \gamma(K_{2(n-1)}).$$

Proposition 2.11. *[20] The genus of the commuting graph of the semidihedral group $SD_{2^n} = \langle r, s | r^{2^{n-1}} = s^2 = 1, srs = r^{2^{n-2}-1}\rangle$ where $n \geq 4$, is given by*

$$\gamma(\Gamma_c(SD_{2^n})) = \gamma(K_{2^{n-1}-2}).$$

Proposition 2.12. *[20] The genus of the commuting graph of the projective special linear group $PSL(2, 2^k)$, where $k \geq 2$, is given by*

$$\gamma(\Gamma_c(PSL(2,2^k))) = (2^k+1)\gamma(K_{2^k-1}) + 2^{k-1}(2^k+1)\gamma(K_{2^k-2}) + 2^{k-1}(2^k-1)\gamma(K_{2^k}).$$

Proposition 2.13. *[20] The genus of the commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ (p is a prime), is given by*

$$\gamma(\Gamma_c(GL(2,q))) = \tfrac{q(q+1)}{2}\gamma(K_{(q-1)(q-2)}) + \tfrac{q(q-1)}{2}\gamma(K_{q(q-1)}) + (q+1)\gamma(K_{(q-1)^2}).$$

In view of Remark 2.6 and the results obtained, one can easily compute the genus of the commuting graph of the group $A \times G$, where A is a finite abelian group and G is any one of the groups considered in the Propositions 2.9 to 2.13. We characterize all finite non-abelian groups whose commuting graphs are planar. However, we have the following lemma containing a couple of elementary properties of finite 2-groups.

Lemma 2.14. *[20] Let G be a finite 2-group. Then, the following assertions hold:*

1. *If $|G| \geq 16$, then G contains an abelian subgroup of order 8.*
2. *If $|G| \geq 32$ and $|Z(G)| \geq 4$, then G contains an abelian subgroup of order 16.*

Our next lemma of this section provides some useful information regarding the size of G and its abelian subgroups.

Lemma 2.15. *[20] Let G be a finite non-abelian group whose commuting graph is planar. Then the following assertions hold:*

1. *If p is a prime divisor of $|G|$, then $p \leq 5$.*
2. *Neither 9 nor 25 divides $|G|$, and hence, $|G|$ is even with $|G| \geq 6$.*

Given a finite non-abelian group G, whose commuting graph is planar, it follows from Lemma 2.15 that $|G| = 2^r 3^s 5^t$, where $r \geq 1$ and $s, t \in \{0, 1\}$. However, depending on the values of $|Z(G)|$, the range of possible values of $|G|$ gets reduced further.

Lemma 2.16. *[20] Let G be a finite non-abelian group whose commuting graph is planar. Then the possible values of $|G|$ are given as follows:*

1. *If $|Z(G)| = 1$, then $|G| = 2^r 3^s 5^t$, where $1 \leq r \leq 3$ and $s, t \in \{0, 1\}$.*
2. *If $|Z(G)| = 2$, then $|G| \in \{8, 12, 24\}$.*
3. *If $|Z(G)| = 4$, then $|G| = 16$.*
4. *$|Z(G)| \neq 3$.*

Note that some of the possibilities mentioned in Lemma 2.16 are not maintainable: for example, in (1), it is obviously not possible to have $s = t = 0$. In fact, the following small result helps us in avoiding few more finite groups as far as the planarity of their commuting graphs is concerned.

Lemma 2.17. *[20] Let G be a finite non-abelian group. If $|G| = 30$, or if G is a solvable group with $|G| = 60$ or 120, then G has an subgroup of order 15 (which is obviously abelian). Also, if $|G| = 40$, then G has an abelian subgroup of order 10.*

In view of Lemma 2.3(3) and Lemma 2.16, it follows from Lemma 2.17 that if G is a finite non-abelian group whose commuting graph is planar, then $|G| \notin \{30, 40\}$; in addition, if G is solvable, then $|G| \notin \{60, 120\}$. We also have the following useful result concerning the groups of order 16.

Lemma 2.18. *[20] Let G be a finite non-abelian group with $|Z(G)| = 4$. Then, the commuting graph of G is planar if and only if $|G| = 16$.*

Remark 2.19. *[20] Up to isomorphism, there are exactly six non-abelian groups of order* 16 *with centers of order* 4*, namely, the two direct products* $\mathbb{Z}_2 \times D_8$ *and* $\mathbb{Z}_2 \times Q_8$*, the Small Group* $SG(16,3) = \langle a, b | a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$*, the semi-direct product* $\mathbb{Z}_4 \times \mathbb{Z}_4 = \langle a, b | a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$*, the central product* $D_8 \times \mathbb{Z}_4 = \langle a, b, c | a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2cb \rangle$ *and the modular group* $M_{16} = \langle a, b | a^8 = b^2 = 1, bab = a^5 \rangle$.

We now state the main result, where two new groups make their appearance, namely, the Suzuki group $Sz(2) = \langle a, b | a^5 = b^4 = 1, bab^{-1} = a^2 \rangle$, and the special linear group $SL(2,3) = \langle a, b, c | a^3 = b^3 = c^2 = abc \rangle$.

Theorem 2.20. *[20] Let* G *be a finite non-abelian group. Then, the commuting graph of* G *is planar if and only if* G *is isomorphic to either* $S_3, D_{10}, A_4, Sz(2), S_4, A_5, D_8, Q_8, D_{12}, Q_{12}, SL(2,3), \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, SG(16,3), \mathbb{Z}_4 \times \mathbb{Z}_4, D_8 \times \mathbb{Z}_4$ *or* M_{16}.

We characterize all finite non-abelian groups whose commuting graphs are toroidal. The following result is analogous to Lemma 2.15.

Lemma 2.21. *[20] Let* G *be a finite non-abelain group whose commuting graph is toroidal. Then, the following assertions hold:*

1. $|Z(G)| \leq 3$.
2. *If* p *is a prime divisor of* $|G|$*, then* $p \leq 7$.
3. *None of* $25, 27$ *and* 49 *is a divisor of* $|G|$.

Analogous to Lemma 2.15, we have the following result concerning the groups of order 16.

Lemma 2.22. *[20] Let* G *be a finite non-abelian* 2*-group with* $|Z(G)| = 2$*. Then, the commuting graph of* G *is toroidal if and only if* $|G| = 16$*, that is, if and only if* G *is isomorphic to either* D_{16}, Q_{16} *or* SD_{16}.

We also have the following result concerning the finite groups that are not 2-groups.

Lemma 2.23. *[20] Let* G *be a finite non-abelian group with* $|G| = 2^r m$*, where* $r \geq 0$*,* $m > 1$ *and* m *is odd. If the commuting graph of* G *is toroidal, then* $r \leq 3$.

If G is a finite non-abelian group whose commuting graph is toroidal, then it follows from Lemma 2.21 that $|G| = 2^r 3^s 5^t 7^u$, where $r \geq 0, 0 \leq s \leq 2$ and $t, u \in \{0, 1\}$. However, as in Lemma 2.16, the range of possible values of $|G|$ gets reduced further depending on the values of $|Z(G)|$.

Lemma 2.24. *[20] Let* G *be a finite non-abelian group whose commuting graph is toroidal. Then the possible values of* $|G|$ *are given as follows:*

1. *If* $|Z(G)| = 1$*, then* $|G| = 2^r 3^s 5^t 7^u$ *where* $0 \leq r \leq 3$ *and* $s, t, u \in \{0, 1\}$.
2. *If* $|Z(G)| = 2$*, then* $|G| \in \{16, 24\}$.
3. *If* $|Z(G)| = 3$*, then* $|G| = 18$.

Needless to mention that some of the possibilities mentioned in Lemma 2.24 are clearly not maintainable; for example, in (1), it is impossible to have $s = t = u = 0$, $r = u = 0$ or $r = s = 0$. Moreover, in view of Lemma 2.3(3) and Lemma 2.24, it follows from Lemma 2.17 that if G is a finite non-abelian group whose commuting graph is toroidal, then $|G| \notin \{30, 40\}$; in addition, if G is solvable, then $|G| \notin \{60, 120\}$. The following result, along with Lemma 2.17, helps us in rejecting some more possibilities.

Lemma 2.25. *[20] Let G be a finite non-abelian group whose commuting graph is toroidal. If $|G| = 7m$, where $m \geq 2$ and $7 \nmid m$, then $m = 2$ or 3.*

We now state the main result of finite non-abelian groups whose commuting graphs are toroidal.

Theorem 2.26. *[20] Let G be a finite non-abelian group. Then, the commuting graph of G is toroidal if and only if G is isomorphic to either $D_{14}, \mathbb{Z}_7 \times \mathbb{Z}_3, \mathbb{Z}_2 \times A_4, \mathbb{Z}_3 \times S_3, D_{16}, Q_{16}$ or SD_{16}.*

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Survey on clean index of ring and nil clean index of ring

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Abstract. In this article we have presented the development on clean index of ring and nil clean index of ring.

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1 Introduction

Ring theory is a board area of mathematics, in the study of ring usually one associate some property **P** to the elements of the ring, then study the class of ring with or without the property the **P**, e.g. of **P** : commutative under multiplication.

Throughout this survey unless or otherwise explicitly stated R will denote an associative ring with unity. We will use the symbols $\mathrm{vnr}(R), \mathrm{Nil}(R), \mathrm{U}(R)$ and $\mathrm{Idem}(R)$ respectively to denote the set of all Von Neumann regular elements, nilpotent elements, units and idempotents of R. Also $J(R)$ will denote the Jacobson radical of R.

In 1936, Von Neumann defined that an element defined that an element $a \in R$ is regular or Von Neumann regular if $a = aba$ for some $b \in R$. Similarly an element $a \in R$ is called unit regular if $a = aua$ for some $u \in \mathrm{U}(R)$ or equivalently $a = eu$ for some $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$. In 1939, McCoy generalized Von Neumann regular rings to π-regular rings, a ring in which for each element $a \in R$ its some positive integral power is Von Neumann regular i.e., for each $a \in R$ there exists an element x and a positive integer n such that $a^n = a^n x a^n$. R is π-regular if all its elements are so. Again an element $a \in R$ is said to be strongly π-regular if there exists an element x and a positive integer such that $a^n = a^{n+1}x$ (Azumaya called such an element right π-regular and similarly he defined left π-regular, further he called an element strongly π-regular if it is both left and right π-regular). Later on F. Dischinger showed that this definition is right and left symmetric. R is strongly π-regular if all its elements are so. In 1977, W.K. Nicholson [10], introduced the notion of clean element, element that can be expressed as a sum of an idempotent and a unit.

Clearly the concept of clean element is an additive analogue of unit regular element. A ring is called a clean ring if each of its element is clean. A ring R is called strongly clean if every $a \in R$ can be expressed as $a = e + u$, where $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$ with $eu = ue$. A.J. Diesl [6] in his Ph.D thesis introduced the concept of nil clean rings, which is a ring in which every element of R can be expressed as a sum of an element of $\mathrm{Idem}(R)$ and an element of $\mathrm{Nil}(R)$. It is easy to see that a nil clean ring is always a clean ring and of course the converse is not true.

As nil clean rings is a sub class of clean rings we started reading about properties of clean rings. Very often, a ring-theoretic property of an element f in a ring S give rise to an interesting description in the case when S is realized as the ring of endomorphisms of some module M_R.

To study the connection between the above mention class of rings, we go through paper [3] where ring elements are treated as endomorphism of some module.

Lemma 1.1. *Let $S = \mathrm{End}(M_R)$ and $f, e \in S$, where e is an idempotent with $A = Ker(e)$ and $B = Im(e)$. Then $f - e \in U(S)$ if and only if there exists an $R-$module decomposition $M = C \oplus D$ such that $f(A) \subseteq C, (1-f)(B) \subseteq D$ and both $f : A \to C$ and $1 - f : B \to D$ are isomorphisms.*

Proof. First suppose that $f = e + u$ for some unit $u \in S$. Put $C = uA$ and $D = uB$. As $f(1-e) = (e+u)(1-e) = u(1-e)$ and $(1-f)e = (e-f)e = -ue$, Thus $f : A \to C$ and $1 - f : B \to D$ are isomorphisms.
Conversely, suppose there is a decomposition $M = C \oplus D$ such that $f : A \to C$ and $1-f : B \to D$ are isomorphisms.Now the same equations as above show that $u := f - e$ is an isomorphism from A to C and from B to D. Thus, $u \in U(S)$. □

By rewriting above lemma we get definition of clean element in the ring $\mathrm{End}(M_R)$.

Proposition 1.2. *An element $f \in \mathrm{End}(M_R)$ is clean if and only if there exist $R-$module decompositions $M = A \oplus B = C \oplus D$ such that $f(A) \subseteq C, (1-f)(B) \subseteq D$, and both $f : A \to C$ and $1 - f : B \to D$ are isomorphisms.*

Such a decomposition of module M will be referred to as an $ABAB$ decomposition for M. Can be represented in diagram as follows.

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & f \downarrow \simeq & & (f-1) \downarrow \simeq \\ M & = & C & \oplus & D \end{array}$$

Similarly we get condition for $f \in \mathrm{End}(M_R)$ to be strongly clean as in the following proposition.

Proposition 1.3. *An element $f \in \mathrm{End}(M_R)$ is strongly clean if and only if there exist $R-$module decompositions $M = A \oplus B$ such that $f(A) \subseteq A, (1-f)(B) \subseteq B$, and both $f : A \to A$ and $1 - f : B \to B$ are isomorphisms.*

$ABAB$ decomposition is given by

$$\begin{array}{ccccc} M & = & A & \oplus & B \\ & & f \downarrow \simeq & & (1-f) \downarrow \simeq \\ M & = & A & \oplus & B \end{array}$$

In [12] Nicholson has given many characterization of strongly $\pi-$regular and strongly regular element of $\mathrm{End}(M_R)$, we take take following.

Proposition 1.4. *Let R be a ring, and let M_R be a right R-module.*
1. An element $f \in \mathrm{End}(M_R)$ is strongly $\pi-$regular if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are $f-$invariant and such that $f|_A \in \mathrm{End}(A)$ is an isomorphism and $f|_B \in \mathrm{End}(B)$ is nilpotent.

2. An element $f \in \mathrm{End}(M_R)$ is strongly regular if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are $f-$invariant and such that $f|_A \in \mathrm{End}(A)$ is an isomorphism and $f|_B \in \mathrm{End}(B)$ is zero.

For rings, ring elements, as well as module endomorphisms, above Propositions show a clear hierarchy of the following four notions:

strongly regular $\Rightarrow$ strongly $\pi-$regular $\Rightarrow$ strongly clean $\Rightarrow$clean. $\cdots(i)$

In his PhD thesis, Alexander James Diesl,[6] introduce similar type of characterization of strongly nil clean element of $\mathrm{End}(M_R)$.

Proposition 1.5. *Let R be a ring, and let M_R be a right $R-$module. An element $f \in \mathrm{End}(M_R)$ is called strongly nil clean if and only if there exists a direct sum decomposition $M = A \oplus B$ such that A and B are $f-$invariant and such that $f|_A \in \mathrm{End}(A)$ is nilpotent and $(1-f)|_B \in \mathrm{End}(B)$ is nilpotent.*

From (i) and above we get following hierarchy:
strongly nil clean $\Rightarrow$ strongly $\pi-$regular $\Rightarrow$ strongly clean $\Rightarrow$ clean.

Huanyin Chen [5] characterize the strongly nil cleanness of 2×2 matrices over local rings. For commutative local rings, he characterize strongly nil cleanness in terms of solvability of quadratic equations. The strongly nil cleanness of a single triangular matrix was describe as well.

Huanyin Chen [4] characterize the strongly nil cleanness of matrices over projective-free rings in terms of the factorizations of their characteristic polynomials.

Myung-Sook Ahn And D.D. Anderson [2] introduce weakly clean rings. A ring R is weakly clean for $a \in R$, there exist some $e \in \mathrm{Idem}(R)$ and $u \in \mathrm{U}(R)$, such that if $a = u + e$ or $u - e$. And prove that if R is weakly clean and $\mathrm{Idem}(R) = \{0, 1\}$ then R is not clean has exactly two maximal ideals and $2 \in U(R)$.

2 Clean index of a ring.

As stated in the abstract this is a survey on various indexes related to clean/nil-clean rings, we start with clean index of rings.

In [14, 15] Tsiu-Kwen Lee and Yiqiang Zhou introduced clean index of a ring. For $a \in R$, let $\xi(a) = \{\ e \in R : e^2 = e, a - e \in \mathrm{U}(R)\ \}$ where $U(R)$ is the group of units of R and the clean

index of R, denoted in(R), is defined by in$(R) = \sup\{|\xi(a)| : a \in R\}$, and they characterize the (arbitrary) rings of clean indices 1, 2, 3,...,7 and determine the abelian rings of finite clean index. Here we state some of the important results.

BASIC PROPERTIES

Some basic properties related to clean index are:

Lemma 2.1. ***(Lemma 1. [14])***
Let R be a ring and let $e, a, b \in R$. Then the following hold:

1. *If $e \in R$ is a central idempotent or a central nilpotent, then $|\xi(e)| = 1$, so $in(R) \geq 1$.*
2. *If $a - b \in J(R)$, then $\xi(a) = \xi(b)$.*
3. *Let σ be an automorphism or anti-automorphism of R. Then $e \in \xi(a)$ iff $\sigma(e) \in \xi(\sigma(a))$; so $|\xi(a)| = |\xi(\sigma(a))|$. In particular, $|\xi(a)| = |\xi(uau^{-1})$ for each $u \in U(R)$.*
4. *If a ring R has at most n units or at most n idempotents, then $in(R) \leq n$. In particular, if R is a local ring then $in(R) \leq 2$.*
5. *If R is local, then $in(R) \leq 2$ iff $R/J(R) \ncong \mathbb{Z}_2$.*
6. *Let R be a clean ring with $2 \in U(R)$. Then $in(R) = |\xi(2^{-1})|$, which is the number of all idempotents of R.*

Lemma 2.2. ***(Lemma 2. [14])***
If S is a subring of a ring R where S and R may or may not share the same identity, then $in(S) \leq in(R)$.

Lemma 2.3. ***(Lemma 3. [14])***
Let $R = S \times T$ be the direct product of two rings S and T. Then $in(R) = in(S)in(T)$.

CHARACTERIZATION OF RINGS WITH CLEAN INDICES

Following theorem characterize the rings of clean index 1. A ring is called abelian if each of its idempotents is central.

Theorem 2.4. ***(Theorem 5. [14])***
$in(R) = 1$, if and only if R is abelian and for any $0 \neq e^2 = e \in R, e \neq u+v$ for non zero $u, v \in U(R)$.

A ring R is called an elemental ring if the idempotents of R are trivial and $1_R \neq u + v$ for some $u, v \in U(R)$. Examples of elemental rings include the rings R and $R[x]$, where R is any local ring with $R/J(R) \ncong \mathbb{Z}_2$.

Theorem 2.5. *$in(R) = 2$ iff one of the following holds:*

1. *R is an elemental ring.*
2. *$R = A \times B$ where A is an elemental ring and $in(B) = 1$).*

3. $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $in(A) = in(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 2$.

The abelian rings of finite clean index are determined in the next theorem. This result is also useful to characterize the rings of clean index larger than 2.

Theorem 2.6. ***(Theorem 15. [14])***
Let R *be an abelian ring with* $in(R) < \infty$ *and let* $k \geq 1$. *The following statements hold:*

1. $in(R) = 2^k$ *iff either* $R = R_1 \times \cdots \times R_k$ *or* $R = S \times R_1 \times \cdots \times R_k$, *where* $in(S) = 1$ *and each* R_i *is an elemental ring.*

2. $p \nmid in(R)$ *for any odd prime* p.

Theorem 2.7. ***(Lemma 19. [14])***
Let A *and* B *be rings and* ${}_AM_B$ *a nontrivial bimodule. If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ *is a formal triangular matrix ring, then* $in(A) < in(R)$ *and* $in(B) < in(R)$.

The rings of clean index 3 are determined by following result.

Theorem 2.8. ***(Theorem 24. [14])***
$in(R) = 3$ *iff* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $in(A) = in(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$.

Following results on clean indices are interesting and was used to derived in the characterization of rings with clean index 4.

Theorem 2.9. ***(Lemma 4. [15])***
Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* A *and* B *are rings,* ${}_AM_B$ *is a bimodule. Let* $n = in(A)$ *and* $m = in(B)$. *The following hold:*

1. $in(R) \geq |M|$.

2. *If* $(M, +) \cong C_{p^k}$, *where* p *is a prime and* $k \geq 1$, *then* $in(R) \geq n + [\frac{n}{2})(|M| - 1)$ *where* $[\frac{n}{2})$ *denotes the least integer greater than or equal to* $n/2$;

3. *Either* $in(R) \geq nm + |M| - 1$, *or* $in(R) \geq 2nm$.

Lemma 2.10. ***(Lemma 5. [15])***
Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* A *and* B *are rings,* ${}_AM_B$ *is a bimodule with* $(M, +) \cong C_{2^r}$. *Then* $in(R) = 2^r in(A) in(B)$.

Formal triangular matrix ring with clean index 4, is caracterized in the followig result.

Theorem 2.11. ***(Proposition 6. [15])***
Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* A *and* B *are rings,* ${}_AM_B$ *is a non trivial bimodule. Then* $in(R) = 4$ *if and only if one of the following holds:*

1. $(M,+) \cong C_2$ *and* $in(A)in(B) = 2$.

2. $(M,+) \cong C_4$ *and* $in(A) = in(B) = 1$.

3. $(M,+) \cong C_3$ *plus one of the following:*

   (a) A *is an elemental ring and* $in(B) = 1$*;*

   (b) $in(A) = 1$ *and* B *is an elemental ring;*

   (c) $in(B) = 1, A = S \times T$ *where* $in(S) = 1$ *and* T *is an elemental ring, and* $SM = 0$*; so* $R \cong S \times \begin{pmatrix} T & M \\ 0 & B \end{pmatrix}$.

   (d) $in(A) = 1, B = S \times T$ *where* $in(S) = 1$ *and* T *is an elemental ring, and* $TM = 0$*; so* $R \cong \begin{pmatrix} A & M \\ 0 & S \end{pmatrix} \times T$.

4. $(M,+) \cong C_2 \oplus C_2$ *plus one of the following*

   (*a*) $in(A) = in(B) = 1$.

   (*b*) $in(A) = 1, \quad B = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *where* $in(S) = in(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$, *for all* $e^2 = e \in A$ *and* $f \in \xi(b)$, *where* $b \in B$ *with* $|\xi(b)| = 2$.

   (*c*) $in(B) = 1, \quad A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, *where* $in(S) = in(T) = 1$ *and* $|W| = 2$, *and* $eM(1_B - f) + (1_A - e)Mf \neq 0$, *for all* $e^2 = e \in B$ *and* $f \in \xi(a)$, *where* $a \in A$ *with* $|\xi(a)| = 2$.

Full characterization of rings with clean index for is in the following theorem'

Theorem 2.12. ***(Theorem 8. [15])***
Let R *be a ring. Then* $in(R) = 4$ *iff one of the following holds:*

1. $R \cong A \times B$ *where* $in(A) = in(B) = 2$*;*

2. R *is isomorphic to a formal triangular matrix ring of clean index* 4 *as given in Proposition*

3. $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where* $in(A) = in(B) = 1, |M| = |N| = 2, MN \subseteq J(A)$, *and* $NM \subseteq J(B)$*;*

4. $R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$, *where* $A = \begin{pmatrix} S & V \\ 0 & T \end{pmatrix}$, *with* $in(S) = in(T) = 1, |V| = 2, in(B) = 1, |M| = |N| = 2$, *and* $MN = 0 = NM$. *Moreover, for any* $e \in \xi(a)$ *where* $a \in A$ *with* $|\xi(a)| = 2$ *and any* $f^2 = f \in B, eM(1-f) + (1-e)Mf \neq 0$, *or* $fN(1-e) + (1-f)Ne \neq 0$.

For characterization of rings with clean indices $5, 6, 7$ readers can see [15].

3 NIL CLEAN INDEX

In this section we survey the results of the notion of nil clean index of a ring and characterization of arbitrary rings with nil clean index 1 and 2. Also few results for rings with indices 3 and 4 are listed [11].

For an element $a \in R$, if $a - e \in \text{nil}(R)$ for some $e^2 = e \in R$, then $a = e + (a - e)$ is called a nil clean expression of a in R and a is called a nil clean element. The ring R is called nil clean if each of its elements is nil clean. A ring R is uniquely nil clean if every element of R has a unique nil clean expression in R. For an element a of R, we denote

$$\eta(a) = \{e \in R \mid e^2 = e \text{ and } a - e \in \text{nil}(R)\}$$

and nil clean index of R, denoted by $Nin(R)$ is defined as

$$Nin(R) = \sup\{\mid \eta(a) \mid : a \in R\}$$

where $\mid \eta(a) \mid$ denotes the cardinality of the set $\eta(a)$. Thus R is uniquely nil clean if and only if R is a nil clean ring of nil clean index 1.

Elementary properties

Some basic properties related to nil clean index are presented here for the study on nil clean index of ring. In this section we have reproduced few proofs which are bid interesting.

Lemma 3.1. *Let R be a ring and let $e, a, b \in R$. Then the following hold:*

1. *If $e \in R$ is a central idempotent or a central nilpotent, then $\mid \eta(e) \mid = 1$, so $Nin(R) \geq 1$.*
2. *$e \in \eta(a)$ iff $1 - e \in \eta(1 - a)$ and so $\mid \eta(a) \mid = \mid \eta(1 - a) \mid$.*
3. *If $f : R \to R$ is a homomorphism, then $e \in \eta(a)$ implies $f(e) \in \eta(f(a))$ and for the converse f must be a monomorphism.*
4. *If a ring R has at most n idempotents or at most n nilpotent elements, then $Nin(R) \leq n$.*

Lemma 3.2. *If S is a subring of a ring R, where S and R may or may not share the same identity, then $Nin(S) \leq Nin(R)$.*

Lemma 3.3. *Let $R = S \times T$ be the direct product of two rings S and T. Then $Nin(R) = Nin(S)Nin(T)$.*

Proof. Since S and T are subrings of R, so

$$\text{Nin}(S) \leq \text{Nin}(R) \text{ and } \text{Nin}(T) \leq \text{Nin}(R).$$

If $\text{Nin}(S) = \infty$ or $\text{Nin}(T) = \infty$, then $\text{Nin}(R) = \infty$ and hence, $\text{Nin}(R) = \text{Nin}(S)\text{Nin}(T)$ holds. Now let

$$\text{Nin}(S) = n < \infty \text{ and } \text{Nin}(T) = m < \infty.$$

As $n,\ m \geq 1$ and there exist elements $s \in S$ and $t \in T$, such that

$$| \eta_S(s) | = n \text{ and } | \eta_T(t) | = m.$$

Thus $s = e_i + n_i$, for $i = 1, 2, \ldots, n$ and $t = f_j + m_j$ for $j = 1, 2, ..., m$, where e_i's, f_j's are idempotents and n_i's, m_j's are nilpotent elements of S and T respectively. Therefore $(s, t) \in R$, can be expressed as

$$(s, t) = (e_i, f_j) + (n_i, m_j),$$

which are mn nil clean expressions of $(s, t) \in R$. Hence

$$\text{Nin}(R) \geq mn.$$

If possible let $\text{Nin}(R) > nm$, say $nm + 1$, then there exists an element $(a, b) \in R$, such that it has at least $nm + 1$ nil clean expressions in R. That is

$$(a, b) = (g_i, h_i) + (c_i, d_i),$$

where $i = 1, 2, \ \ldots, mn+1, \ \ (g_i, h_i)^2 = (g_i, h_i)$ and $(c_i, d_i) \in \text{nil}(R)$. So $a = g_i + c_i$ and $b = h_i + d_i$ are nil clean expressions for a and b respectively. Let

$$K = \{(g_i, h_i) \mid i = 1, \ \ 2, \ \ 3, \ldots, mn, \ \ mn + 1\}.$$

Now we have

$$\begin{aligned} & | K | = nm + 1 \\ \Rightarrow \quad & | \{g_i\} | \, . \, | \{h_i\} | = nm + 1 \\ \Rightarrow \quad & | \{g_i\} | > n \text{ or } | \{h_i\} | > m, \end{aligned}$$

which gives $\text{Nin}(S) > n$ or $\text{Nin}(T) > m$, which is absurd. □

Lemma 3.4. *Let I be an ideal of R with $I \subseteq nil(R)$ and let $n \geq 1$ be an integer. Then the following hold*

1. *If idempotents lift modulo I, then $Nin(R/I) = NinR$.*
2. *If $Nin(R) \leq n$, then every idempotent of R/I can be lifted to at most n idempotents of R.*

Lemma 3.5. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ is a bimodule. Let $Nin(A) = n$ and $Nin(B) = m$. Then*

1. *$Nin(R) \geq |M|$.*
2. *If $(M, +) \cong C_{p^k}$, where p is a prime and $k \geq 1$, then $Nin(R) \geq n + [\frac{n}{2}](|M| - 1)$, where $[\frac{n}{2})$ denotes the least integer greater than or equal to $\frac{n}{2}$.*
3. *Either $Nin(R) \geq nm + |M| - 1$ or $Nin(R) \geq 2nm$.*

Proof. (i) Let $\alpha = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}$. Then we have

$$\left\{ \begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} \mid w \in M \right\} \subseteq \eta(\alpha)$$

as

$$\begin{pmatrix} 1_A & w \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \text{ is nilpotent.}$$

So we have

$$\mathrm{Nin}(R) \geq |\eta(\alpha)| \geq |M|.$$

(ii) Let $q = p^k$ and $a = e_i + n_i,\ \ i = 1, 2, \ldots n$ be n distinct nil clean expressions of a in A. For any $e = e^2 \in A$

$$(M, +) = eM \oplus (1 - e)M.$$

Since $(M, +) \cong C_{p^k}$, so $(M, +)$ is indecomposable and hence

$$M = eM \text{ or } = (1 - e)M.$$

Assume that

$$(1 - e_1)M = \cdots = (1 - e_s)M = M \text{ and } e_{s+1}M = \cdots = e_nM = M.$$

If $s \geq (n - s)$ (i.e., $s \geq [\frac{n}{2}]$), then for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & 0 \end{pmatrix}$, we have

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1_A - e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, 1 \leq j \leq s, 0 \neq w \in M \right\},$$

so

$$|\eta(\alpha)| \geq n + s(q - 1).$$

If $s < (n - s)$ (i.e., $n - s \geq [\frac{n}{2}]$), for $\beta = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$

$$\eta(\beta) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e_j & w \\ 0 & 0 \end{pmatrix} : 1 \leq i \leq n, s + 1 \leq j \leq n, 0 \neq w \in M \right\},$$

therefore

$$|\eta(\beta)| \geq n + (n - s)(q - 1).$$

Hence

$$\mathrm{Nin}(R) \geq n + \left\lceil \frac{n}{2} \right\rceil (q - 1).$$

(iii) Let $a = e_i + n_i,\ \ i = 1, 2, \ldots n$ and $b = f_j + m_j,\ \ j = 1, 2, \ldots m$ be distinct nil clean expressions of a and b in A and B respectively.

Case I:

If $e_{i_0}M(1 - f_{j_0}) + (1 - e_{i_0})Mf_{j_0} = 0$ for some i_0 and j_0. Then $e_{i_0}w = wf_{i_0}$ for all $w \in M$. Thus for $\alpha = \begin{pmatrix} 1_A - a & 0 \\ 0 & b \end{pmatrix}$

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} 1_A - e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} 1_A - e_{i_0} & w \\ 0 & f_{j_0} \end{pmatrix}; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w \in M \right\},$$

so we have $|\eta(\alpha)| \geq mn + |M| - 1$.
Case II:
If $e_iM(1-f_j) + (1-e_i)Mf_j \neq 0$ for all i and j. Take

$$0 \neq w_{ij} \in e_iM(1-f_j) + (1-e_i)Mf_j \text{ for each pair } (i,j).$$

For $\alpha = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, we have

$$\eta(\alpha) \supseteq \left\{ \begin{pmatrix} e_i & 0 \\ 0 & f_j \end{pmatrix}, \begin{pmatrix} e_i & w_{ij} \\ 0 & f_j \end{pmatrix}; 1 \leq i \leq n, 1 \leq j \leq m; 0 \neq w_{ij} \in M \right\},$$

thus $|\eta(\alpha)| \geq 2mn$.
Cases I and **II** imply, either

$$\text{Nin}(R) \geq nm + |M| - 1 \text{ or } \text{Nin}(R) \geq 2nm.$$

□

Lemma 3.6. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ is a bimodule with $(M,+) \cong C_{2^r}$. Then $Nin(R) = 2^r Nin(A)Nin(B)$.*

Lemma 3.7. *Let A and B be rings and ${}_AM_B$ a nontrivial bimodule. If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ is a formal triangular matrix ring, then $Nin(A) < Nin(R)$ and $Nin(B) < Nin(R)$.*

Lemma 3.8. *Let R be a ring with unity, then $In(R) \geq Nin(R)$, where $In(R)$ is the clean index of R.*

Proof. Let $\text{Nin}(R) = k$, then there is at least an element $a \in R$, such that it has k nil clean expressions in R, i.e.,

$$a = e_i + n_i, \text{ for } i = 1, 2, \cdots, k,$$

where $e_i \in \text{idem}(R)$ and $n_i \in \text{nil}(R)$. From this we get,

$$a - 1 = e_i + (n_i - 1)$$

are k clean expression for $(a-1) \in R$, and therefore $\text{In}(R) \geq k$, hence for any arbitrary ring with unity

$$In(R) \geq Nin(R).$$

□

Rings of nil clean indices 1, 2 and 3

Theorem 3.9. *$Nin(R) = 1$ if and only if R is an abelian ring.*

Proof. ($\Rightarrow$) Part is obvious.
($\Leftarrow$) Let R be an abelian ring and e a non zero idempotent of R. We claim that e can not be written as sum of two nilpotent elements. Suppose $e = a + b$ where $a, b \in \mathrm{Nil}(R)$ and for positive integers $n < m$, $a^n = 0 = b^m$. Then $(e-a)^m = 0$, using binomial theorem we get

$$e^m - \binom{m}{1}ae^{(m-1)} + \binom{m}{2}a^2e^{(m-2)} - \cdots + (-1)^{(n-1)}\binom{m}{n-1}a^{(n-1)}e^{(m-n+1)} = 0$$

which gives

$$e\left[1 - \binom{m}{1}a + \binom{m}{2}a^2 - \cdots + (-1)^{(n-1)}\binom{m}{n-1}a^{(n-1)} + (-1)^n\binom{m}{n}a^n + (-1)^{(n+1)}\binom{m}{n+1}a^{(n+1)} + \cdots + (-1)^m a^m\right] = 0$$

this implies

$$e(1-a)^m = 0,$$

therefore we get, $e = 0$ [since $1 - a \in \mathrm{U}(R)$].

Similarly, if $n > m$, then $(e-b)^n = 0$ and so $e = 0$, a contradiction. Hence, no nonzero idempotent can be written as sum of two nilpotent elements and therefore $\mathrm{Nin}(R) = 1$. □

Above theorem gives following observations:

1. A ring R with $\mathrm{Nin}(R) = 1$ is always Dedekind finite, but the converse is not true can be verified with counter example.

2. Rings with trivial idempotents have nil clean index one and consequently the local rings are of nil clean index one. If $\mathrm{Nin}(R) = 1$, then it is easy to see that idempotents of $R[[x]]$ are idempotents of R, and for any

$$\alpha = \alpha_0 + \alpha_1 x + \cdots \in R[[x]],$$

we have

$$\eta_{R[[x]]}(\alpha) \subseteq \eta_R(\alpha_0),$$

this gives

$$\mathrm{Nin}(R[x]) = \mathrm{Nin}(R[[x]]) = 1.$$

But if $\mathrm{Nin}(R) > 1$ then, there is some noncentral idempotent $e \in R$, such that $er \neq re$ for some $r \in R$. So either

$$er(1-e) \neq 0 \text{ or } (1-e)re \neq 0.$$

Let $er(1-e) \neq 0$, then we have

$$\begin{aligned} a &= e + er(1-e) \\ &= [e + er(1-e)x^i] + [er(1-e)(1-x^i)], \end{aligned}$$

where i is a positive integer, are infinitely many nil clean expressions of a in $R[x]$, which implies

$$\mathrm{Nin}(R[x]) = \infty.$$

Now we have the following theorem.

Theorem 3.10. *Nin(R[[x]]) is finite iff Nin(R) = 1.*

We characterize rings of nil clean index 2. From the discussion above we see that such rings must be non abelian.

Theorem 3.11. *Nin(R) = 2 if and only if* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where Nin(A) = Nin(B) = 1 and* ${}_AM_B$ *is a bimodule with* $|M| = 2$.

Proof. ($\Leftarrow$) For $\alpha_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix} \in R$, we have

$$\left\{ \begin{pmatrix} 0 & \omega \\ 0 & 1_B \end{pmatrix};\ \ \omega \in M \right\} \subseteq \eta(\alpha_0).$$

Therefore

$$Nin(R) \geq | \eta(\alpha_0) | \geq | M | = 2.$$

For any $\alpha = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in R$

$$\eta(\alpha) = \left\{ \begin{pmatrix} e & w \\ 0 & f \end{pmatrix};\ e \in \eta(a),\ f \in \eta(b),\ w = ew + wf \right\}.$$

Because $| M | = 2$, $| \eta(a) | \leq 1$, $| \eta(b) | \leq 1$, it follows that $| \eta(\alpha) | \leq 2$, hence $\mathrm{Nin}(R) = 2$.
($\Rightarrow$) Suppose R is non abelian and let $e^2 = e \in R$ be a non central idempotent. If neither $eR(1-e)$ nor $(1-e)Re$ is zero, then take $0 \neq x \in eR(1-e)$ and $0 \neq y \in (1-e)Re$. Then

$$\begin{aligned} e &= e + 0 \\ &= (e + x) - x \\ &= (e + y) - y \end{aligned}$$

are three distinct nil clean expressions of e in R. So without loss of generality, we can assume that

$$eR(1-e) \neq 0 \text{ but } (1-e)Re = 0.$$

The Peirce decomposition of R gives

$$R = \begin{pmatrix} eRe & eR(1-e) \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

As above $2 = \mathrm{Nin}(R) \geq | eR(1-e) |$; so $| eR(1-e) | = 2$. Write

$$eR(1-e) = \{0, x\}.$$

If possible let $a = e_1 + n_1 = e_2 + n_2$ be two distinct nil clean expressions of a in eRe. If $e_1 x = x$

$$\begin{aligned} \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & x \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 0 \end{pmatrix} \end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R$. If $e_1 x = 0$,

$$\begin{aligned}\begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix} &= \begin{pmatrix} e_1 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_2 & 0 \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_2 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e_1 & x \\ 0 & 1_B \end{pmatrix} + \begin{pmatrix} n_1 & x \\ 0 & 1_B \end{pmatrix}\end{aligned}$$

are three distinct nil clean expressions of $\begin{pmatrix} a & 0 \\ 0 & 1_B \end{pmatrix}$ in R. This contradiction shows that $\mathrm{Nin}(eRe) = 1$, similarly, $\mathrm{Nin}((1-e)R(1-e)) = 1$. □

The next proposition gives a sufficient condition for rings to have nil clean index 3.

Proposition 3.12. *If $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where $Nin(A) = Nin(B) = 1$ and ${}_AM_B$ is a bimodule with $|M| = 3$, then $Nin(R) = 3$.*

Next we have following proposition for matrix ring.

Proposition 3.13. *Let S be a ring with unity and let $n \geq 2$ be an integer. Then*

1. *$Nin(M_n(S)) \geq 3$.*
2. *$Nin(M_n(S)) = 3$ iff $n = 2$ and $S \cong \mathbb{Z}_2$.*

Formal triangular ring with nil clean index 4

Theorem 3.14. *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, where A and B are rings, ${}_AM_B$ is a non trivial bimodule. Then $Nin(R) = 4$ if and only if one of the following holds:*

1. *$(M, +) \cong C_2$ and $Nin(A)Nin(B) = 2$.*
2. *$(M, +) \cong C_4$ and $Nin(A) = Nin(B) = 1$.*
3. *$(M, +) \cong C_2 \oplus C_2$ plus one of the following*
   (a) *$Nin(A) = Nin(B) = 1$.*
   (b) *$Nin(A) = 1$, $B = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, where $Nin(S) = Nin(T) = 1$ and $|W| = 2$, and $eM(1_B - f) + (1_A - e)Mf \neq 0$, for all $e^2 = e \in A$ and $f \in \eta(b)$, where $b \in B$ with $|\eta(b)| = 2$.*
   (c) *$Nin(B) = 1$, $A = \begin{pmatrix} S & W \\ 0 & T \end{pmatrix}$, where $Nin(S) = Nin(T) = 1$ and $|W| = 2$, and $eM(1_B - f) + (1_A - e)Mf \neq 0$, for all $e^2 = e \in B$ and $f \in \eta(a)$, where $a \in A$ with $|\eta(a)| = 2$.*

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Finite fields in the process of compute convolutions

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Abstract. In this chapter, we are trying to analyse the applications of finite fields to compute convolutions of finite sequences of integers. To do so, we are using basics of finite field of the form $\mathbb{F}_{p^n}$. Furthermore, d-point Fourier-like transforms are introdcued and proven to be the only linear transforms in $\mathbb{F}_{p^n}$ which has the circular convolution property. It is well known that the set $\mathfrak{G}(p)$ of integers modulo p (a prime) is subfield of $\mathbb{F}_{p^n}$, then the d-point transformation is used over $\mathbb{F}_{p^n}$ to compute the transform of a sequence of integers $\{a_1, a_2, \ldots, a_n, \}$ where a_n lies in the range $-[(p-1)/2] \leq a_n \leq (p-1)/2$. Thus, for two such sequences, by using the circular convolution computed using d-point transforms over $\mathbb{F}_{p^n}$. Again the d-point transforms of $\mathbb{F}_{q^n}$ are shown to be perfect for computing convolutions of two sequences of complex integers also. The number of points in the transform, should divide the order $q^2 - 1 = 2^{p+1}(2^{P-1} - 1)$ of the multiplicity subgroup of $\mathbb{F}_{p^2}$, so the number of points in a circular transform over $\mathbb{F}_{p^2}$ can be chosen to be a power of 2. Thus one of the best method is the fast Fourier transform (FFT) algorithm to compute convolutions of complex numbers without any kind of round-off error.

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Keywords. finite field, discrete Fourier transform.

1 Introduction

M.Rader in [1] already proven that the convolution of two finite sequences of integers (a_n) and (b, n) for $n = 1, 2, ..., d$ can be obtained as the inverse transform of the product of two transforms which must be other than the usual discrete Fourier transform (DFT). Rader defined transforms of the form

$$\mathfrak{A}_n = \sum_{i=0}^{d-1} a_i 2^{ni} \quad mod(M)$$

. [1]

where M is either a Mersenne prime number $M = 2^p - 1$, p be a prime , or M was the Fermat number

$$M = 1 + 2^{2^k},$$

where k an integer.

One of the primary advantages of the Rader transform over the DFT,

$$\mathfrak{F}_n = \sum_{i=0}^{d-1} a_i \omega^{in}$$

[2]

where $w\omega$ is a d root of unity, which using by the logic that the multiplications by powers of ω can be replaced in binary arithmetic by simple shifts. Obviously , this advantage must be weighed against the difficulties of computing the answer $modulo(M)$ and of the numeric constraints, relating word length, length of sequence d together with compositeness of d, added by the above choices for M, introduced by Rader. This chapter is focused on review of the Rader transform first by studying the class of transforms, given by equation [1], proceeded by presenting more information of the computational algorithm for calculating such a convolution with [1]. Next, the class of transforms in [1] will increase to include a Fourier type transform over a finite field $\mathbb{F}_q$. This generalization was discussed by J. M. Pollard in [2].

2 Discrete Fourier Transformation on Finite Field

The number of elements finite field $\mathbb{F}_q$ is of the form $q = p^n$, in which p is prime and n is positive integer. To construct a finite field $\mathbb{F}_q$, first we have to consider an n degree irreducible polynomial $f(x)$ over $\mathbb{F}_q$. The elements of $\mathbb{F}_q$ are all in polynomial form

$$g(\alpha) = \sum_{i=0}^{n-1} a_i \alpha^i$$

where a_i's are in $\mathbb{F}_q$.where α is a root of $pf(x)$, i.e., $f(\alpha) = 0$. Then, the product $g(\alpha)$ of two elements of $\mathbb{F}_q$ say $g_1(\alpha)$ and $g_2(\alpha)$ in $\mathbb{F}_q$ be the residue of $g_1(x)g_2(x)modf(x)$ with α substituted for x. That is, $g(\alpha)$ is found by

$$g(x) \equiv g_1(x)g_2(x)modf(x)$$

where $x = \alpha$. Similarly, the addition is defined as $S(\alpha)$ is found by

$$S(x) = g_1(x) + g_2(x)modf(x)$$

where $x = \alpha$. By taking addition and multiplication of all polynomials $f(\alpha)$ in this form, the addition and multiplication is defined. For further details readers are requested to see [3].

Let α and x be elements of a finite field, denoted by $\mathbb{F}_q$, and then we consider the mapping of subset of d distinct non-zero elements

$$\Delta_d = \{\chi_0, \chi_1, \cdots \chi_{d-i}\} \in \mathbb{F}_q$$

.

into $\mathbb{F}_q$ with the following mapping

$$\Lambda(x) = \sum_{i=0}^{d-1} \alpha_i x^i$$

[3]

which is the most general possible mapping from $\mathbb{F}_q$ into $\mathbb{F}_q$. This mapping can also be displayed as a system of linear equations in which the coefficients are in $\mathbb{F}_q$ and which can be represented as matrix form.

$$\Lambda_d = \chi\lambda_d$$

.

Where Λ_d and λ_d are column matrices.

$$\text{Where } \lambda_d = \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ . \\ . \\ . \\ \alpha_{d-1} \end{bmatrix} \text{ and } \Delta_d = \begin{bmatrix} \Lambda(\chi_0) \\ \Lambda(\chi_1) \\ . \\ . \\ . \\ \Lambda(\chi_{d-1}) \end{bmatrix}$$

$$\text{and } \chi = \begin{bmatrix} 1 & \chi_0 & \chi_0^2 & \cdots & \chi_0^{d-1} \\ 1 & \chi_1 & \chi_1^2 & \cdots & \chi_1^{d-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & \chi_{d-1} & \chi_{d-1}^2 & \cdots & \chi_{d-1}^{d-1} \end{bmatrix}$$

This polynomial mapping is also denoted as a linear mapping of the vector λ_d into the vector Λ_d. Such a mapping is one to one or is invertible if matrix χ has an inverse, that is, if the determinant $|\chi|$ of χ is non-zero.

Which is determined as

$$|\chi| = \prod_{j<i}(\chi_i - \chi_j)$$

since all the χ_i are distinct, hence determinant is non zero and hence χ^{-1} exists and system of equations can be solved as

$$\lambda_d = \chi^{-1}\Lambda_d$$

,

which is the inverse transformation.

From this onwards we are going to focus on circular convolutions. Next let us apply this on matrix form as the constraint that it can be used to compute circular convolution C_n of sequences a_n and b_n.

$$C_n = \sum_{i=0}^{d-1} a_i b_{n-i}$$

[4]

where $(n-i)$ be the residue of $(n-i) mod\, d$.

main objective is to transform of C_n , namely, C to be given by

$$C = \begin{bmatrix} C(\chi_0) \\ C(\chi_1) \\ \vdots \\ C(\chi_{d-1}) \end{bmatrix} = \begin{bmatrix} \Lambda(\chi_0) \,.\, \beta(\chi_0) \\ \Lambda(\chi_1) \,.\, \beta(\chi_1) \\ \vdots \\ \Lambda(chi_{d-1}) \,.\, \beta(\chi_{d-1}) \end{bmatrix} = \Lambda \otimes \beta$$

Then after expanding, equating the coefficients of $a_j b_k$, one gets

$$\chi_i^{(j+k)} = \chi_i^{j+k}$$

[5]

for $(k, j, i = 0, 1, 2...d-1)$ where $(j+k)$ being the residue of $(j+k)$ mod d.

Now for transform to yield circular convolutions, χ_i, must be a d-th root of unity for $i = 1, 2, ...d$ in $\mathbb{F}_q$.

Since the group $\mathbb{F}_q^*$ is cyclic group of order $q-1$, the application for χ for an element $\chi_i \in \mathbb{F}_q$ gives that integer d divides $q-1$. i.e, $d|q-1$ then the transform (3) will yield a circular convolution. Furthermore, since the set of elements $(\chi_0, \chi_1, ..., \chi_{d-1})$ are distinct and all are dth roots of unity. Hence this set is also a cyclic subgroup of the cyclic subgroup of $\mathbb{F}_q^*$. Hence the set, $(\chi_0, \chi_1, ..., \chi_{d-1})$, equals the subgroup $\{1, \alpha, \alpha^2, ..., \alpha^{d-1}\} = \psi_d$ i.e.,

$$(\chi_0, \chi_1, ..., \chi_{d-1}) = \{1, \alpha, \alpha^2, ..., \alpha^{d-1}\} = \psi_d$$

[6]

where α is a primitive element of $\mathbb{F}_q^*$.

We consider that the group $\{1, \alpha, \alpha^2, ..., \alpha^{d-1}\} = \psi_d$ is replaced for $(\chi_0, \chi_1, ..., \chi_{d-1})$ in transform (3), the transform becomes

$$\Lambda_i = \sum_{j=0}^{d-1} a_i \alpha^{ij}$$

[7]

where $i = 0, 1, 2, \ldots, d-1$.

For invert transform, we are using the fact that all the elements of ψ_d are dth root of unity, ie they satisfy the equation

$$x^d - 1 = 0$$

.

Now $x^d - 1$ factors as following

$$x^d - 1 = (x-1)\sum_{j=0}^{d-1} x^j$$

[8].

Then we have the following

$$\sum_{j=0}^{d-1} x^j = 0 \,\text{for}\, x \neq 1 \,\text{and}\, x \in \psi_d.$$

$$\sum_{j=0}^{d-1} x^j = 1 = [d] \,\text{for}\, x = 1.$$

Where $[d]$ is d mod$p$. This formula is introduced by Pollard [4] and by Reed and Solomon in [5].

Now for inverse convolution, we need discrete delta function as given below

$$\delta_d(m) = 0 \text{ if } m \not\equiv 0 \, mod \, d$$
$$= 1 \text{ if } m \equiv 0 \, mod \, d$$

Then we have the inverse transformation I_j for $j = 0, 1, 2, \ldots, d-1$ as follows

$$[d]^{-1} \sum_{j=0}^{d-1} I_j \alpha^{-jp} = \sum_{i=0}^{d-1} a_i b_{(p-i)}$$

[9]

The result in [9], gives that satisfy the conditions on the transform, given by [3], is both necessary and sufficient for transform [3] to computing circular convolutions. This generalizes a similar theorem, given by Agarwal and Burrus in [6], for the field of complex numbers i.e, $\mathbb{C}$ to all fields both finite and infinite. Now, we are going to focus on how to restrict the finite field transform, given by [8], so that it yields circular convolutions over both the integers and complex integers.

FINITE FIELD TRANSFORMS

If n is an integer of magnitude such that less than or equal $(p-l)/2$ where p is a prime. Then integer a satisfies

$$-[(p-1)/2] \leq n \leq (p-1)/2$$

If $n > 0$,then n is the residue mod p.
If $n < 0$ i.e, $n = -m$, where $m > 0$, then $n \equiv p - m$ mod$p$

Thus the set of positive integers

$$\{-\frac{p-1}{2}, \ldots, -2, -1, 0, 1, 2, \ldots, \frac{p-11}{2}\}$$

relates in a one-to-one manner with the following set of residues mod p,

$$\{(p - \frac{p-1}{2}), \ldots, (p-2), (p-1), 0, 1, 2, \ldots, \frac{p-11}{2}\}$$

Since the latter one consists of all residues modulo p, this set uniquely represents the set of all positive and negative real integers of magnitude less than or equal to $(p-1)/2$. Furthermore, this set of residues mod p composes precisely the finite field $\mathbb{F}_p$, hence the above maps the set of integers less than or equal to $(p-1)/2$ onto $\mathbb{F}_q$ in a one-to-one way. To do the arithmetic operations in $\mathbb{F}_p$ which arrive at the correct arithmetic answer, one must often restrict the operating ranges of the integer variables even further.

To determine convolution, let a and b are elements in a finite field $\mathbb{F}_p$ with transforms of the form suggested by Rader [1], we need first introduce the integers in such a field. To preserve the arithmetic operations of addition and multiplication on $\mathbb{F}_p$, the representation must necessarily restricted to $\mathbb{F}_p$ as shown . Moreover, $\mathbb{F}_p$ is a subfield of $\mathbb{F}_{p^n}$; in fact, the prime field of $\mathbb{F}_p$ for all $n = 1, 2, 3, ...$ Thus, convolutions can be computed with transforms as shown above on a finite field $\mathbb{F}_{p^n}$ where a and b are restricted to $\mathbb{F}_p$.

If an element α can be determine in $\mathbb{F}_q$ so that multiplications by powers of α are simple in calculation, the extension might be handy in increasing the number of possible points in the convolution. This immediately follows from the fact that d is a divisor of $p^n - 1$ and the number of divisors of $p^n - 1$ is obviously greater than the number of divisors of $p - 1$.

Furthermore it can be determined that , for certain prime numbers p, this computational requirement can be reduced from four to two Rader-type transforms. For that, the following condition on p is necessary.

$$X^2 - 1 \equiv 0 \bmod p \text{ is not solvable}$$

Further this is equivalent to

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$$

Where $\left(\frac{a}{b}\right)$ is Legendre's symbol, which is given by

$$\begin{aligned}\left(\frac{a}{b}\right) &= +1, \text{if a quadratic residue mod b} \\ &= -1 \text{ if a is quadratic nonresidue mod b}\end{aligned}$$

Then the two cases emerges on the basis of two kinds of primes.

$$\begin{aligned}&\text{Case 1 =Mersenne primes of the form } q = 2^p - 1 \\ &\text{Case 2 = Format primes of form } q = 2^{2^n} + 1, \text{ where } 1 \leq n \leq 4\end{aligned}$$

From simple calculation, one can determine that the Mersenne primes have an advantage over the Fermat primes in the calculation of convolutions of complex integers over $\mathbb{F}_p$. Inspite of that, Rader points in [1] states that, this advantage must be weighed against the fact that the fast Fourier transform (FFT) algorithm can be applied to the transforms, using Fermat primes, but not by using the Mersenne primes.

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Review on t-core Partition Function

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Abstract. Study of different partition function with some certain restriction has got great attention in recent times.One of the interesting partition function is t-core partition. A partition of n is called a t-core partition of n if none of its hook number is divisible by t. Here, in this paper, we give the brief introduction of t-core partition and we have done a literature review about the work done on the t-core partition function.

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Keywords. t-core partition; partition congruence; q-series identities; Ramanujan's theta-functions.

1 Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum equals n. The number of partitions of a positive integer n is denoted by $p(n)$. For convenience, we set $p(0) = 1$, which means it is considered that 0 has one partition. The generating function for the partition function is generally given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} \tag{1.1}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n). \tag{1.2}$$

In 1919, Ramanujan [12], [13, p. 210-213] established

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.3}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.4}$$

$$p(11n+6) \equiv 0 \pmod{11} \tag{1.5}$$

Ramanujan's theta-functions identities are defined by

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty (q^2;q^2)_\infty (q;q^2)_\infty (-q^2;q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = (q^2;q^2)_\infty (q;q^2)_\infty,$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty,$$

where $f(a,b) = \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}$, $|ab| < 1$ is the Ramanujan's general theta-function

Motivated by Ramanujan's congruences on $p(n)$ many other partition function are studied and Ramanujan type congruences are established by several mathematicians and researchers.One of the famous partition function is t-core partition. For a given partition $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ the Ferrers-Young diagram of λ is an array of nodes with λ_i nodes in the ith row. The (i,j) hook is the set of nodes directly below, together with the set of nodes directly to the right of the (i,j) nodes, as well as the (i,j) nodes itself.The hook number is the total number of nodes on the (i,j) hook. A partition of n is called a t-core partition of n if none of its hook number is divisible by t.

Example:

The Ferrers-Young diagram of the partition 3+2+1 of 6 is

The nodes (1, 1), (1, 2), (1, 3), (2, 1), (2, 2) and (3, 1) have hook numbers 5, 3, 1, 3, 1 and 1, respectively. It is easily seen from above that the partition $3 + 2 + 1$ of 6 is 4-core but it is not a 3-core and 5-core.

The number of partitions of n that are t-cores is denote by $a_t(n)$. From [11, Eq.(2.1)], the generating function $a_t(n)$ is given by

$$\sum_{n=0}^{\infty} a_t(n) q^n = \frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}. \tag{1.6}$$

2 Review of related literature for t-core partition function

In recent times study of partition function with some certain restrictions has become one of the popular research topic. The arithmetic properties of t-core partitions have been studied by many authors, for example see [2, 3, 5, 6, 7, 8, 8, 9, 19] and references there in. Example of a 3-core identity also appears in Baruah and Berndt [1]. Numerous congruences of the t-core partition function have been established in the spirit of Ramanujan by employing theta function identities and modular equations.

Hirchhorn and Sellers [19] proved two congruences involving 4-cores for modulo 2 and 4 respectively.

Theorem 2.1. *[19] For all $n \geq 0$, we have*

$$a_4(9n+2) \equiv 0 \pmod 2, \tag{2.1}$$

$$a_4(9n+8) \equiv 0 \pmod 4. \tag{2.2}$$

Hirchhorn ans Sellers [7] proved many amazing facts about 4-cores. For example,

Theorem 2.2. *[7] For $n \geq 0$, we have*

$$a_4(27n+5) = 3a_4(3n), \tag{2.3}$$

$$a_4(27n+14) = 5a_4(3n+1). \tag{2.4}$$

Hirchhorn ans Sellers [8] derived an explicit formula for $a_3(n)$ by using elementary means in terms of the prime factorization of $3n+1$. Chen [8] proved a conjecture on congruences for 2^t-core partition and also established many congruences for p-core partition function when $5 \leq p \leq 47$. Baruah and Nath [2] proved some results for 3-core partition function.

Theorem 2.3. *[2] If $u(n)$ denotes the number of representations of a non-negative integer n in the form x^2+3y^2 with $x, y \in Z$, and $a_3(n)$ is the number of 3-cores of n, then*

$$u(12n+4) \equiv 6a_3(n). \tag{2.5}$$

Lemma 2.4. *[2] If $u(n)$ denotes the number of representations of a non-negative integer n in the form x^2+3y^2 with $x, y \in Z$ and if $p \equiv 2 \pmod 3$ is an odd prime, then*

$$u(p^2n) = u(n). \tag{2.6}$$

Baruah and Nath [3] also proved some results for 4-core partition function. By employing theta function identities they proved that $a_4(8n+5) = 8a_4(n) = v(8n+5) = \frac{1}{3}r_3(8n+5)$, where $u(n)$ and $v(n)$ are number of representations of a nonnegative integer n in the forms $x^2+4y^2+4z^2$ and $x^2+2y^2+2z^2$, where $x, y, z \in Z$ and $a_4(n)$ and $r_3(n)$ are the number of 4-cores of n and the number of representations of n as a sum of three squares, respectively. Baruah and Nath [3] also established some infinite family of arithmetic relations of $a_4(n)$. For example,

Theorem 2.5. *[3] For $n \geq 0$, and $k \geq 1$,*

$$\Big(\frac{5^{k+1}-1}{4}\Big)a_4(25n) = a_4\Big(5^{2k+2}n + \frac{5^{2k+1}-5}{8}\Big). \tag{2.7}$$

Theorem 2.6. *[3] For $n \geq 0$, and $k \geq 1$,*

$$\Big(\frac{5^{k+1}-1}{4}\Big)a_4(25n+5) = a_4\Big(5^{2k+2}n + \frac{9\cdot 5^{2k+1}-5}{8}\Big). \tag{2.8}$$

Theorem 2.7. *[3] For $n \geq 0$, and $k \geq 1$,*

$$\Big(\frac{5^{k+1}-1}{4}\Big)a_4(25n+10) = a_4\Big(5^{2k+2}n + \frac{17\cdot 5^{2k+1}-5}{8}\Big). \tag{2.9}$$

Theorem 2.8. *[3] For $n \geq 0$, and $k \geq 1$,*

$$\Big(\frac{5^{k+1}-1}{4}\Big)a_4(25n+20) = a_4\Big(5^{2k+2}n + \frac{33\cdot 5^{2k+1}-5}{8}\Big). \tag{2.10}$$

3 Concluding Remarks

As we have seen that there are so many congruences and identities for t-core partition function that have been proved by different mathematicians. Those results that are obtained by employing modular equation or q-series identities which will help the researchers to do futher research in partition theory in the spirit of Ramanujan.

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Generalized Frobenius Partitions with k colors

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Abstract. In this chapter, we give an introduction to generalized Frobenius partitions with k colors. We also give a brief review of literature on the works done so far on the generating functions and congruences for generalized Frobenius partitions with k colors for different values of k.

2010 Mathematical Sciences Classification. Primary 11P83; Secondary 05A15, 05A17

Keywords. Partitions; Generalized Frobenius partition; partition congruence; generating function.

1 Introduction

In 1984, Andrews [1] introduced generalized Frobenius partitions (or, simply F-partitions). A generalized Frobenius partition or an F-partition of n is a notation of the form

$$\begin{pmatrix} a_1 & a_2 & . & . & . & a_r \\ b_1 & b_2 & . & . & . & b_r \end{pmatrix}$$

of nonnegative integers a_i's, b_i's with

$$n = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i,$$

where each row is of the same length and each is arranged in non-increasing order. Andrews considered two general classes of F-partitions, in one of which each non-negative integer is allowed to have k-copies (colors) and strict decrease in each row is maintained. If $c\phi_k(n)$ denotes the number of such F-partitions of n, then the generating function for $c\phi_k(n)$ is

$$\sum_{n=0}^{\infty} c\phi_k(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^k} \sum_{m_1,m_2,\cdots,m_{k-1}=-\infty}^{\infty} q^{Q(m_1,m_2,\cdots,m_{k-1})},$$

where $|q| < 1$ and

$$Q(m_1, m_2, \cdots, m_{k-1}) = \sum_{j=1}^{k-1} m_j^2 + \sum_{1\le i<j\le k-1} m_i m_j.$$

2 Literature Review

Andrews [1] found the generating functions

$$\sum_{n=0}^{\infty} c\phi_1(n)q^n = \frac{1}{(q;q)_\infty},$$
$$\sum_{n=0}^{\infty} c\phi_2(n)q^n = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty^4(q^4;q^4)_\infty},$$
$$\sum_{n=0}^{\infty} c\phi_3(n)q^n = \frac{(q^{12};q^{12})_\infty(q^6;q^{12})_\infty^3}{(q;q^6)_\infty^5(q^5;q^6)_\infty^5(q^4;q^4)_\infty^2(q^3;q^6)_\infty^7} + 4q\frac{(q^{12};q^{12})_\infty(q^4;q^4)_\infty}{(q^6;q^{12})_\infty(q^2;q^4)_\infty(q;q)_\infty^3},$$

and proved the congruences

$$c\phi_2(5n+3) \equiv 0 \pmod 5,$$
$$c\phi_k(n) \equiv 0 \pmod{k^2} \quad \text{if } k \text{ is prime and does not divide } n,$$
$$c\phi_k(n) \equiv c\phi_1(n/k) \pmod{k^2} \quad \text{if } k \text{ is prime and divides } n,$$

where, as customary, for any complex number a and $|q| < 1$, we define

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k).$$

In 1987, Kolitsch [14] stated this congruence in a slightly different way and show that it can be extended to include all positive values of m. Specifically, he showed that the number of F-partitions of n using k colors whose order is k under cyclic permutation of the k colors,

$$\overline{c\phi_k}(n) = \sum_{d|(k,n)} \mu(d)c\phi_{k/d}(n/d) \equiv 0 \pmod{k^2}.$$

In [15], he established a relationship between colored F-partitions with 5 and 7 colors and ordinary partitions by proving that for a positive integer n,

$$\overline{c\phi_5}(n) = 5p(5n-1),$$
$$\overline{c\phi_7}(n) = 7p(7n-2),$$

where $p(n)$ denotes the number of partitions of n. In 1990, Kolitsch [16] showed that

$$\overline{c\phi_3}(n) = c\phi_3(n) - p(n/3) = \frac{9q(q^9;q^9)_\infty^3}{(q;q)_\infty^3(q^3;q^3)_\infty}. \tag{2.1}$$

Using this result he then showed that $\overline{c\phi_3}(n)$ is congruent to zero modulo large powers of 3 for certain values of n. Specifically, if λ_α is the reciprocal of 8 modulo 3^α, then

$$\overline{c\phi_3}(3^\alpha n + \lambda_\alpha) \equiv 0 \begin{cases} \pmod{3^{2\alpha+2}}, & \text{if } \alpha \text{ is even,} \\ \pmod{3^{2\alpha+1}}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

For more works by Kolitsch on this partition function, see [17].

Sellers [22] then found the following congruences.

Theorem 2.1. *For $m = 5, 7$ and 11, and for all $n \geq 1$,*

$$\overline{c\phi_m}(mn) \equiv 0 \ (mod\ m^3).$$

He also remarked that congruences like the above might hold for other values of m as well. In another paper, Sellers [23] conjectured that for all $n \geq 1$ and $\alpha \geq 1$,

$$\overline{c\phi_2}(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}, \tag{2.2}$$

where λ_α is the least positive reciprocal of 12 modulo 5^α. Sellers [24] also found the following generating function for $\overline{c\phi_2}(n)$.

Theorem 2.2. *For all $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{c\phi_2}(n)q^n = \frac{4q(q^{16};q^{16})_\infty^2}{(q;q)_\infty^2(q^8;q^8)_\infty}.$$

With the help of the above theorem and (2.1), he proved that

$$\begin{aligned} \overline{c\phi_2}(2n) &\equiv 0 \pmod{2^3}, \\ \overline{c\phi_3}(3n) &\equiv 0 \pmod{3^4}. \end{aligned} \tag{2.3}$$

In 1996, Ono [5] used the theory of modular forms to arrive at the following congruences.

$$\begin{aligned} c\phi_3(63n + 50) &\equiv 0 \pmod 7, \\ c\phi_3(5n + 2) &\equiv p\left(\frac{5n+3}{2}\right) \pmod 5, \end{aligned}$$

except when $n = 3T_m$ and $T_m = \dfrac{m(m+1)}{2}$ is the $m-$th triangular number, and

$$c\phi_3(15T_m + 2) \equiv (-1)^m(m + 3) \pmod 5.$$

Using a technique similar to Ono [5], Lovejoy [19] also proved the following congruences.

$$\begin{aligned} c\phi_3(45n + 23) &\equiv 0 \pmod 5, \\ c\phi_3(45n + 41) &\equiv 0 \pmod 5, \\ c\phi_3(63n + 50) &\equiv 0 \pmod 7, \\ c\phi_3(99n + 95) &\equiv 0 \pmod{11}, \\ c\phi_3(171n + 50) &\equiv 0 \pmod{19}. \end{aligned}$$

The cases $\alpha = 1, 2, 3, 4$ of (2.2) were proved by Eichhorn and Sellers [9] in 2002. However, Sellers' conjecture (2.2), for all α, was proved by Paule and Radu [18] using modular functions that belong to a Riemann surface of genus 1.

Motivated by a question of Lovejoy [19], Xiong [25], using modular forms, proved the following congruences modulo powers of 5.

$$c\phi_3(45n+23) \equiv 0 \pmod{5^4},$$
$$c\phi_3(45n+41) \equiv 0 \pmod{5^4},$$
$$c\phi_3(75n+22) \equiv 0 \pmod{5^2},$$
$$\overline{c\phi_3}(75n+72) \equiv 0 \pmod{5^2}.$$

In [4], Baruah and Ojah found a simple proof of Sellers' result (2.3) and the congruence

$$c\phi_3(3n+2) \equiv 0 \pmod{3^3}.$$

In [2, 3], Baruah and Sarmah applied the integer matrix exact covering systems developed by Cao [5] to find the following generating functions.

Theorem 2.3. *If, for* $|q| < 1$,

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \textit{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

then

$$\sum_{n=0}^{\infty} c\phi_4(n)q^n = \frac{1}{(q;q)_\infty^4}\big(\varphi^3(q^2) + 12q\varphi(q^2)\psi^2(q^4)\big),$$
$$\sum_{n=0}^{\infty} c\phi_5(n)q^n = \frac{1}{(q;q)_\infty^5}\big(\varphi(q^{10})\varphi^3(q^2) + 12q\varphi(q^{10})\varphi(q^2)\psi^2(q^4) + 8q\psi(q^5)\psi^3(q)$$
$$+ 12q^3\psi(q^{20})\psi(q^4)\varphi^2(q) + 16q^4\psi(q^{20})\psi^3(q^4)\big),$$
$$\sum_{n=0}^{\infty} c\phi_6(n)q^n = \frac{1}{(q;q)_\infty^6}\big(\varphi^3(q)\varphi(q^2)\varphi(q^6) + 24q\psi^3(q)\psi(q^2)\psi(q^3) + 4q^2\varphi^3(q)\psi(q^4)\psi(q^{12})\big),$$
$$\sum_{n=0}^{\infty} \overline{c\phi_4}(n)q^n = 16q\frac{\psi^2(q^2)\psi(q^4)}{(q;q)_\infty^4}.$$

Baruah and Sarmah [2, 3] also derived the congruences

$$c\phi_4(2n+1) \equiv 0 \pmod{4^2},$$
$$c\phi_4(4n+3) \equiv 0 \pmod{4^4},$$
$$c\phi_4(4n+2) \equiv 0 \pmod{4},$$
$$c\phi_6(2n+1) \equiv 0 \pmod{4},$$
$$c\phi_4(3n+1) \equiv 0 \pmod{3^2},$$
$$c\phi_6(3n+2) \equiv 0 \pmod{3^2},$$
$$\overline{c\phi_4}(2n) \equiv 0 \pmod{4^3},$$
$$\overline{c\phi_4}(4n+3) \equiv 0 \pmod{4^4},$$
$$\overline{c\phi_4}(4n) \equiv 0 \pmod{4^4}.$$

Working on the generating functions for $c\phi_4(n)$, $c\phi_5(n)$, and $c\phi_6(n)$ given in the above theorem, several other congruences have been found by various authors.

Sellers [19] used the generating function for $c\phi_4(n)$ to obtain the unexpected congruence

$$c\phi_4(10n+6) \equiv 0 \pmod 5.$$

Employing some theta identities due to Ramanujan, the $(p,k)-$parametrization of theta functions and the generating function for $c\phi_4(n)$ given by Baruah and Sarmah [2], Xia [24] proved that

$$c\phi_4(20n+11) \equiv 0 \pmod 5.$$

Hirschhorn and Sellers [13] significantly extended the study of congruences satisfied by $c\phi_4$ modulo 5. By employing classical results in q-series, the well-known theta functions of Ramanujan, and elementary generating function manipulations, they proved a characterization of $c\phi_4(10n+1)$ modulo 5 which leads to an infinite set of Ramanujan-like congruences modulo 5 satisfied by $c\phi_4$. In particular, they proved the following:

Theorem 2.4. *For any nonnegative integer n,*

$$c\phi_4(10n+1) \equiv \begin{cases} k+1 \pmod 5, & \text{if } n=k(3k+1) \text{ for some integer } k, \\ 0 \pmod 5, & \text{otherwise.} \end{cases}$$

Lin [18] also used the generating function for $c\phi_4(n)$ given by Baruah and Sarmah [2] to discover a Ramanujan-type congruence modulo 7 for 4-colored generalized Frobenius partition function. He proved that for $n \geq 0$,

$$c\phi_4(14n+13) \equiv 0 \pmod 7.$$

Zhang and Wang [29] extended the study of congruences satisfied by $c\phi_4$ modulo 7. By employing Ramanujan's congruence $p(7n+5) \equiv 0 \pmod 7$ and a corollary of the quintuple product identity, proved that for all $n \geq 0$,

$$c\phi_4(7n+6) \equiv 0 \pmod 7.$$

While work on Ramanujan-like congruence properties satisfied by the functions $c\phi_k(n)$ continues, unfortunately, in all cases, the authors restrict their attention to small values of k. This is often due to the difficulty in finding a "nice" representation of the generating function for $c\phi_k(n)$ for large k. Because of this, no Ramanujan-like congruences are known where k is large. Garvan and Sellers [10] rectified this situation by proving several infinite families of congruences for $c\phi_k(n)$ where k is allowed to grow arbitrarily large. The proof is truly elementary, relying on a generating function representation which appears in Andrews' Memoir [1] but has gone relatively unnoticed. Garvan and Sellers [10] gave the following theorem.

Theorem 2.5. *Let p be prime and let r be an integer such that $0 < r < p$. If*

$$c\phi_k(pn+r) \equiv 0 \pmod p$$

for all $n \geq 0$, then

$$c\phi_{pN+k}(pn+r) \equiv 0 \pmod p$$

for all $N \geq 0$ and $n \geq 0$.

Cui, Gu and Huang [10] obtained many infinite families of congruences for $c\phi_k(n)$, For example, we have the following theorems.

Theorem 2.6. *For $\alpha \geq 1$, $n \geq 0$, and $i = 1, 2, 3, 4$,*

$$c\phi_2\left(2 \cdot 5^{4\alpha} n + \frac{(24i + 55) \cdot 5^{4\alpha-1} + 1}{12}\right) \equiv 0 \pmod{2^3}.$$

For $i = 1, 2, 3, 4, 5, 6$,

$$c\phi_2\left(2 \cdot 7^{4\alpha} n + \frac{(24i + 77) \cdot 7^{4\alpha-1} + 1}{12}\right) \equiv 0 \pmod{2^3}.$$

Theorem 2.7. *For $\alpha \geq 1$, $n \geq 0$, and $i = 1, 2, 3, 4$,*

$$c\phi_4\left(4 \cdot 5^{4\alpha} n + \frac{(24i + 85) \cdot 5^{4\alpha-1} + 1}{6}\right) \equiv 0 \pmod{2^9}.$$

For $i = 1, 2, 3, 4, 5, 6$,

$$c\phi_4\left(4 \cdot 7^{4\alpha} n + \frac{(24i + 119) \cdot 7^{2\alpha-1} + 1}{6}\right) \equiv 0 \pmod{2^9}.$$

For more works on generalized Frobenius partitions with k colors, we refer the papers by Chan, Wang and Yang [6], Xia [23], Hirschhorn [12], and Gu, Wang and Xia [11].

Very recently, Chan, Wang and Yang [7] used the theory of modular forms to find representations of the generating functions for $c\phi_k(n)$ for all positive integers $k \leq 17$. They also found a host of new congruences. For example, they [7, Theorem 5.2] found the following congruences.

Theorem 2.8. *For any nonnegative integer n,*

$$\begin{aligned} c\phi_9(9n + 3) &\equiv 0 \pmod{3^2}, \\ c\phi_9(9n + 6) &\equiv 0 \pmod{3^2}, \\ c\phi_9(3n + 1) &\equiv 0 \pmod{3^4}, \\ c\phi_9(3n + 2) &\equiv 0 \pmod{3^6}. \end{aligned}$$

3 Conclusion

Most of the congruences mentioned above have been proved by using the theory of modular forms. It would be interesting to find the elementary proofs in the spirit of Ramanujan. Also, the exact generating functions for $c\phi_k$ for larger values of k is still unknown. That remains as an open problem for further research.

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Broken k-diamond partition functions and k dots bracelet partition functions

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Abstract. In 2007, Andrew's and Paul introduced broken k-diamond partition functions. After that several mathematician studied the congruence properties for this partition function and in 2011, Fu generalized the broken k-diamond partitons and called it as k dots bracelet partition function. In this chapter, we give a brief note on broken k-diamond partition functions and k dots bracelet partition functions, and also give a brief account of works done so far on these partition functions.

2010 Mathematical Sciences Classification. Primary 11P83; Secondary 05A15, 05A17.

Keywords. Partitions; partition congruences.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of a non-negative integer n is a finite sequence of non-increasing positive integer *parts* λ_i such that $n = \sum_{i=1}^{k} \lambda_i$. The partition function $p(n)$ is the number of partitions of a non-negative integer n, with the convention that $p(0) = 1$. For example, we have $p(4) = 5$, as there are five partitions of 4, namely, (4), $(3,1)$, $(2,2)$, $(2,1,1)$ and $(1,1,1,1)$. The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where, here and throughout the chapter, for $|q| < 1$, $(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n)$.
Ramanujan's so-called "most beautiful identity" for the partition function $p(n)$ is

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}, \tag{0.1}$$

which readily implies one of his three famous partition congruences, namely,

$$p(5n+4) \equiv 0 \pmod 5. \tag{0.2}$$

The other two famous partition congruences found by Ramanujan are

$$p(7n+5) \equiv 0 \pmod 7 \tag{0.3}$$

and

$$p(11n+6) \equiv 0 \pmod{11}. \tag{0.4}$$

We refer to a recent paper by Bruinier, Folsom, Kent and Ono [5] for further references on the partition function.

MacMahon in his renowned book "Combinatory Analysis" [16] introduced the partition analysis as the most important tool for solving combinatorial problems which are related with the system of linear diophantine inequalities and equations. MacMahon commenced with the most simplest case of plane partitions where the non-negative integers a_i of the partitions placed at the corners of a square such that the following order relations are satisfied:

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4 \text{ and } a_3 \geq a_4. \tag{0.5}$$

To represent $\geq$ relation, an arrow can be used as an alternative, for instance Fig.1 represents (0.5). Here and throughout the chapter, an arrow pointing from a_i to a_j is interpreted as $a_i \geq a_j$.

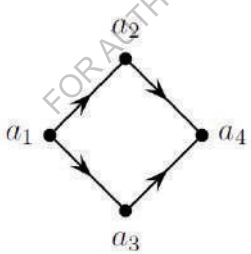

Figure .1: The inequality

By using partition analysis, MacMahon derived the generating function

$$\begin{aligned}\varphi &:= \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}, \\ &= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},\end{aligned}$$

where the sum is taken over all non-negative integers a_i satisfying (0.5). MacMahon also observed that, by putting $x_1 = x_2 = x_3 = x_4 = q$, the generating function becomes

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

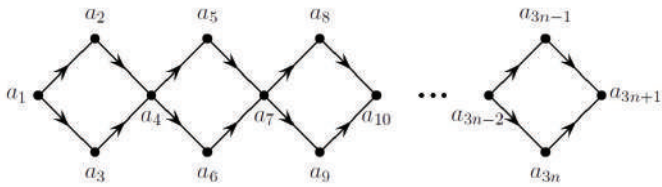

Figure .2: A plane partition diamond of length n

By using MacMahon's partition analysis, Andrews, Paule and Riese [2] introduced partition diamonds as new variations of plane partitions as shown in Fig.2.

In 2007, Andrews and Paule [3] studied the generalization of this partition diamonds by introducing k-elongated partition diamonds as shown in Fig.3, as the building blocks of the chain.

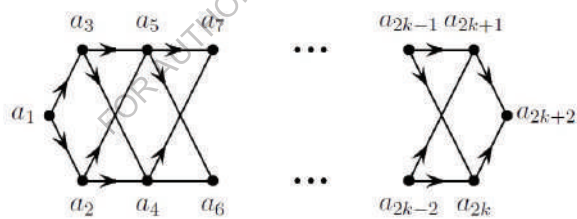

Figure .3: A k-elongated partition diamond of length 1

Andrews and Paule [3] also introduced Broken k-diamonds. Broken k-diamonds consist of two separated k-elongated partition diamonds of length n where in one of them, the source is deleted, as shown in Fig.4.

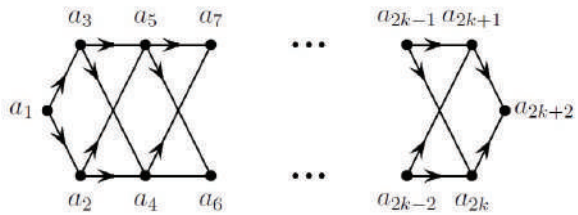

Figure .4: A broken k-diamond of length $2n$

Definition 0.1. *For $n,\ k \geq 1$, define*

$$H^{\diamondsuit}_{n,k} := \{(b_2, \ldots, b_{(2k+1)n+1}, a_1, a_2, \ldots, a_{(2k+1)n}) \in \mathbb{N}^{(4k+1)n},$$
$$\text{the } a_i \text{ and } b_i \text{ satisfy all order relations in Fig.4}\}$$
$$h^{\diamondsuit}_{n,k} := h^{\diamondsuit}_{n,k}(x_2, \ldots, x_{(2k+1)n+1}, y_1, y_2, \ldots, y_{(2k+1)n+1})$$
$$:= \sum_{(b_2,\ldots,b_{(2k+1)n+1},a_1,a_2,\ldots,a_{(2k+1)n}) \in H^{\diamondsuit}_{n,k}} x_2^{b_2} \cdots x_{(2k+1)n+1}^{b_{(2k+1)n+1}} y_1^{a_1} y_2^{a_2} \cdots y_{(2k+1)n+1}^{a_{(2k+1)n+1}}$$

and

$$h^{\diamondsuit}_{n,k}(q) := h^{\diamondsuit}_{n,k}(q, q, \ldots, q).$$

Andrews and Paule [3] also found the generating function for the number of broken k-diamond partitions of n as given in the next theorem.

Theorem 0.2. *Let for $n \geq 0$ and $k \geq 1$, $\Delta_k(n)$ denote the total number of broken k-diamond partitions of n. Then*

$$h^{\diamondsuit}_{\infty,k}(q) = \sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)^3_\infty (-q^{2k+1};q^{2k+1})_\infty}. \tag{0.6}$$

For $k = 1$, they also proved the congruence

$$\Delta_1(2n+1) \equiv 0 \pmod 3 \tag{0.7}$$

and stated three more conjectures.

In 2011, Fu [12] gave a combinatorial proof of (0.7) and also applied the combinatorial approach to generalise the broken k-diamond partitions which he called k dots bracelet partitions. Before defining k dots bracelet partitions, Fu defined infinite bracelet partitions which consist of repeating diamonds and dots with $k-2$ dots between two consecutive diamonds as shown in Fig.5 and we see that an infinite bracelet partitions can be cut into $k-1$ different ways with k dots in half. For any $k \geq 3$, a k dots bracelet partitions consist of $k-1$ different half bracelet as shown in Fig.6.

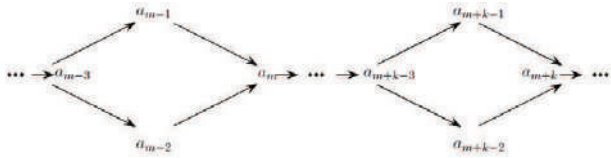

Figure .5: Infinite bracelet with k dots

Fu [12] denoted the number of k dots bracelet partitions for a positive integer n by $\mathfrak{B}_k(n)$. The generating function for $\mathfrak{B}_k(n)$ is given by

$$\sum_{0}^{\infty} \mathfrak{B}_k(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^k(-q^k;q^k)_\infty}. \tag{0.8}$$

He also proved the following congruences for k dots bracelet partitions:

(i) for $n \geq 0, k \geq 3$ if $k = p^r$ is a prime power,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{p},$$

(ii) for any $k \geq 3$, s an integer between 1 and $p-1$ such that $12s+1$ is a quadratic nonresidue modulo p and any $n \geq 0$, if $p|k$ for some prime $p \geq 5$, say $k = pm$, then

$$\mathfrak{B}_k(pn+s) \equiv 0 \pmod{p},$$

(iii) for any $n \geq 0, k \geq 3$ even, say $k = 2^m l$, where l is odd,

$$\mathfrak{B}_k(2n+1) \equiv 0 \pmod{2^m}.$$

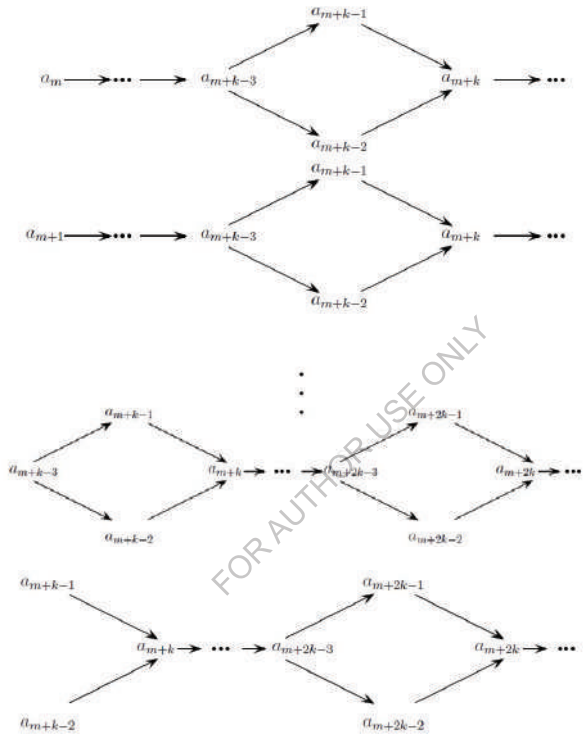

Figure .6: $k-1$ different half bracelet.

1 Review of literature

Hirschhorn and Sellers [11] provided a new proof of (0.7) as well as elementary proofs of congruences modulo 2 for $k = 1$ and 2. Combinatorial proofs of (0.7) were given by Mortenson [17] and Fu [12]. There are a number of other congruences for $\Delta_2(n)$ in [7, 8, 18, 20]. Radu and Sellers [19] found parity results for broken k-diamond partitions for some values of k. Paule and Radu [18] conjectured four congruences for broken 3- and 5-diamond partitions. Two of those congruences were proved by Xiong [25] and the remaining two were proved by Jameson [13]. Radu and Sellers [20] found some parity results for broken 3-diamond partitions by using the theory of modular forms and subsequently, Lin [15] found the elementary proofs of those parity results. Cui and Gu [10]

and Wang [22] also found more parity results for broken 3- and 8-diamond partitions respectively. Recently, Xia [23] found infinite families of congruences modulo 7 for broken 3-diamond partitions.

In [1], Ahmed and Baruah found parity results for broken 5-, 7- and 11-diamond partitions by employing p-dissection of Ramanujan's theta functions. Some of the results are given in the following theorems.

Theorem 1.1. *For any odd prime p, $\alpha \geq 0$ and if n is not a triangular number, then*

$$\Delta_5\left(396 \cdot p^{2\alpha} \cdot n + \frac{99 \cdot p^{2\alpha}+1}{2}\right) \equiv 0 \pmod 2.$$

Theorem 1.2. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\Delta_7\left(8 \cdot 5^{2\alpha+1} \cdot n + 8 \cdot r \cdot 5^{2\alpha} + \frac{16 \cdot 5^{2\alpha}+2}{3}\right) \equiv 0 \pmod 2, \tag{1.1}$$

for $r = 3,\ 4,\ 8,\ 9,\ 13,$ and 14.

Theorem 1.3. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\Delta_{11}(2 \cdot 23^{\alpha+1} \cdot n + 2 \cdot r \cdot 23^{\alpha} + 1) \equiv 0 \pmod 2, \tag{1.2}$$

for $r = 5,\ 7,\ 10,\ 11,\ 14,\ 15,\ 17,\ 19,\ 20,\ 21,\ 22$.

Radu and Sellers [21] extended the set of congruences given by Fu. They proved that for all $n \geq 0$

$$\mathfrak{B}_5(10n+7) \equiv 0 \pmod{5^2},$$
$$\mathfrak{B}_7(14n+11) \equiv 0 \pmod{7^2},$$

and

$$\mathfrak{B}_{11}(22n+21) \equiv 0 \pmod{11^2}.$$

More recently, Cui and Gu [9] found several congruences modulo 2 for 5 dots bracelet partitions and congruences modulo p for any prime $p \geq 5$ for k dots bracelet partitions. Xia and Yao [24] also found several congruences modulo powers of 2 for 5 dots bracelet partitions. Recently, Yao [27] established the generating functions of $\mathfrak{B}_9(An+B)$ modulo 4 for some values of A and B and hence obtained congruences for modulo 2 and 4.

In [4], Ahmed and Baruah found several new congruences modulo 2 for 7 and 11 dots bracelet partitions and also find congruences modulo p^2 and p^3 for k dots bracelet partitions for any prime $p > 3$ by employing Ramanujan's theta functions and by finding the binomial expansion of $(q;q)_\infty^{p^n}$ congruent modulo p^n for $n = 2$ and $n = 3$ respectively. A few results are given below.

Theorem 1.4. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$, where $n \neq \dfrac{k(3k-1)}{2}$, we have*

$$\mathfrak{B}_{11}\left(4 \cdot p^{2\alpha} \cdot n + \frac{p^{2\alpha}+5}{6}\right) \equiv 0 \pmod 2.$$

Theorem 1.5. *Let $k = mp^r$, where $m \in \mathbb{N}$, $p \geq 5$ and $r \geq 2$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn + \ell) \equiv 0 \pmod{p^2},$$

where $1 \leq \ell \leq p-1$ and $12\ell + 1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\frac{12\ell+1}{p}\right) = -1$.

Theorem 1.6. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k(pn + j) \equiv 0 \pmod{p^3},$$

where $1 \leq j \leq p-1$ and $12j + 1$ is quadratic nonresidue modulo p, i.e., in Legendre symbol $\left(\frac{12j+1}{p}\right) = -1$.

Theorem 1.7. *Let $k = mp^s$, where $m \in \mathbb{N}$, $p \geq 5$ and $s \geq 3$. Then for any positive integer n, we have*

$$\mathfrak{B}_k\left(p(pn + j) + \frac{p^2-1}{12}\right) \equiv 0 \pmod{p^2},$$

for $j = 1,\ 2,\ \ldots,\ p-1$.

2 Conclusion

We see that there is a scope of exploring the topics discussed above. In [21], Radu and Sellers used the theory of modular forms to prove the congruences modulo 5^2, 7^2 and 11^2 for 5, 7 and 11 dots bracelet partition functions. So it would be interesting to find some elementary way to prove those congruences, and also to find Ramanujan like identity for these partition functions from which one can generalised the congruences.

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Periodic sequences modulo m

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Abstract. We give a few remarks on the periodic sequence $a_n = \binom{n}{x} \ (mod\ m)$ where $x, m, n \in \mathbb{N}$, which is periodic with minimal length of the period being

$$\ell(m,x) = \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i} x \rfloor + b_i} = m \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i} x \rfloor},$$

where $m = \prod_{i=1}^{w} p_i^{b_i}$. We prove certain interesting properties of $\ell(m,x)$ and derive a few other results and congruences.

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Keywords. prime moduli, binomial coefficients, periodic sequences modulo m.

1 Introduction and Preliminaries

This paper deals with the periodicity of certain binomial coefficients which have previously been studied by many mathematicians, [1] and [6] are some examples of results obtained in this direction. The second author and Vogrinc [4] stated and proved the following theorem.

Theorem 1.1. *A natural number $p > 1$ is a prime if and only if $\binom{n}{p} - \lfloor \frac{n}{p} \rfloor$ is divisible by p for every non-negative n, where $n > p + 1$ and the symbols have their usual meanings.*

The proof of Theorem 1.1 was completed by the present authors [2]. In this section we state without proof the following results which we shall be referring in the coming sections. The proofs can be found in [5].

Definition 1.2. *A sequence (a_n) is said to be periodic modulo m with period k if there exists an integer $N > 0$ such that for all $n > N$*

$$a_{n+k} = a_n \,(\mathrm{mod}\, m).$$

In the following, we shall use usual periodicity with $N = 1$ unless otherwise mentioned.

Theorem 1.3. *The sequence* $(a_n) = \binom{n}{x} \pmod m$ *is periodic, where* $x, m, n \in \mathbb{N}$.

Theorem 1.4. *For a natural number* $m = \prod_{i=1}^{k} p_i^{b_i}$*, the sequence* $a_n \equiv \binom{n}{m} \pmod m$ *has a period of minimal length,*

$$l(m) = \prod_{i=1}^{k} p_i^{\lfloor \log_{p_i} m \rfloor + b_i}.$$

Theorem 1.4 was also derived in [3], however the motivation of that paper was quite different from [5].

The following generalization of Theorem 1.1 was also proved in [2].

Theorem 1.5. *For natural numbers* n, k *and a prime* p *we have the following*

$$\binom{n}{p^k} - \left\lfloor \frac{n}{p^k} \right\rfloor \equiv 0 \pmod p.$$

We also fix the notation $[[1, i]]$ for the set $\{1, 2, \ldots, i\}$ throughout the paper.

Definition 1.6. *We define* $\mathrm{ord}_p(n)$ *for* $n \in \mathbb{N}$ *to be the greatest exponent of* p *with* p *a prime in the decomposition of* n *into prime factors,*

$$\mathrm{ord}_p(n) = \max\left\{k \in \mathbb{N} : p^k | n\right\}.$$

2 Results and Discussion

Remarks on Theorem 1.3

The integer n in Theorem 1.3 should be greater than x. Otherwise, the binomial coefficient $\binom{n}{x}$ is not defined. But, we can extend the definition of $\binom{n}{x}$ to integer n such that $0 \leq n < x$ by setting $\binom{n}{x} = 0$ if $0 \leq n < x$. Nevertheless, notice that this extension is not necessary in order to prove this theorem about periodic sequences.

The case where $m = 0$ is not possible since the sequence $(\binom{n}{x})$ is not periodic modulo 0 or is not simply periodic. So, if $x = m$, x should be non-zero.

If $x = 0$, then we have

$$a_n \equiv a_{n+1} \equiv \ldots \equiv a_{n+k} \equiv 1 \,(\mathrm{mod}\, m)$$

for any integers n and k. So, if $x = 0$, the sequence (a_n) is periodic with minimal period equal to 1. We recall that if a sequence is periodic, a period of such a sequence is a non-zero integer.

In the following, we assume $x \geq 1$.

Lemma 2.1. *For* $n \geq x + 1$

$$\sum_{i=x}^{n-1} \binom{i}{x} = \binom{n}{x+1}.$$

The proof of the above is not difficult and can be done using induction. We omit the details here.

Let k be the length of a period of sequence $a_n \equiv \binom{n}{x} (\mathrm{mod}\, m)$, meaning $\binom{n+k}{x} \equiv \binom{n}{x} (\mathrm{mod}\, m)$. Then we have,

Lemma 2.2.

$$\sum_{j=y}^{y+mk-1} \binom{j}{x} \equiv 0 \,(\mathrm{mod}\, m).$$

Proof. It is enough to notice the following

$$\sum_{j=y}^{y+mk-1} \binom{j}{x} = \sum_{i=0}^{m-1} \sum_{j=y}^{y+k-1} \binom{j+ik}{x} \equiv \sum_{i=0}^{m-1} r = mr \equiv 0 \ (\mathrm{mod}\ m),$$

where $\sum_{j=y}^{y+k-1} \binom{j+ik}{x} \equiv r \,(\mathrm{mod}\, m)$, for some r. □

In [2], the authors mention without proof the following generalization of Theorem 1.4.

Theorem 2.3. *For a natural number $m = \prod_{i=1}^{w} p_i^{b_i}$, the sequence (a_n) such that $a_n \equiv \binom{n}{x} \,(\mathrm{mod}\, m)$ has a period of minimal length*

$$\ell(m, x) = \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i} x \rfloor + b_i} = m \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i} x \rfloor}.$$

The proof follows from the proof of Theorem 1.4 as given in [5] and also via Theorem 3 in [3]. An easy corollary mentioned in [2] is proved below.

Corollary 2.4. *For $m = \prod_{i=1}^{w} p_i^{b_i}$ we have*

$$m^2 \leq \ell(m) \leq m^{w+1}.$$

Proof. We have

$$\ell(m) = m \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i}(m) \rfloor} \geq m \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i}(p_i^{b_i}) \rfloor} = m \prod_{i=1}^{w} p_i^{b_i}$$

and

$$\ell(m) = m \prod_{i=1}^{w} p_i^{\lfloor \log_{p_i}(m) \rfloor} \leq m \prod_{i=1}^{w} p_i^{\log_{p_i}(m)} = m^{w+1}.$$

□

Remark 2.5. *Here $w \leq m - \varphi(m)$ where φ is the Euler totient function.*

We now formally give the following definition.

Definition 2.6 (Minimal Period of a periodic sequence)**.** *The period of minimal length of a periodic sequence (a_n) such that $a_n \equiv \binom{n}{x} \,(\mathrm{mod}\, m)$ with $x \in \mathbb{N}$ and $m \in \mathbb{N}$, is the minimal non-zero natural number $\ell(m, x)$ such that for all positive integer n we have*

$$\binom{n + \ell(m, x)}{x} \equiv \binom{n}{x} \,(\mathrm{mod}\, m)$$

where it is understood that

$$\binom{n}{x} = \begin{cases} 0, & \text{if} \quad 0 \leq n < x \\ \frac{n!}{x!(n-x)!}, & \text{if} \quad n \geq x. \end{cases}$$

Remark 2.7. *If $x = 0$, then $\ell(m, x = 0) = 1$ with $m \in \mathbb{N}$.*

From Definition 2.6

$$\binom{\ell(m,x)}{x} \equiv \binom{\ell(m,x)+1}{x} \equiv \cdots \equiv \binom{\ell(m,x)+x-1}{x} \equiv 0 \,(\mathrm{mod}\, m).$$

If $x > 0$ ($x \in \mathbb{N}$), since any number is divisible by 1, we have

$$\binom{x}{x} \equiv \binom{x+1}{x} \equiv \cdots \equiv \binom{2x-1}{x} \equiv 0 \,(\mathrm{mod}\, 1).$$

Regarding the definition of $\ell(m, x)$, since x is the least non-zero natural number which verifies this property, we can set ($x \in \mathbb{N}$) $\ell(1, x) = 1$.

The minimal period $\ell(m)$ of a sequence (a_n) such that $a_n \equiv \binom{n}{m} \,(\mathrm{mod}\, m)$ with $m \in \mathbb{N}$ (see Theorem 1.4) is given by $\ell(m) = \ell(m, m)$.

Before we mention a few results we recall that $\log_a x = \frac{\ln x}{\ln a}$ and $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$.

Theorem 2.8.

$$\lfloor \log_p(x+1) \rfloor = \left\lfloor \log_p(x) + \frac{1}{x \ln p} \right\rfloor = \begin{cases} \lfloor \log_p(x) \rfloor, & \text{if} \quad x \neq p^c - 1; \\ \lfloor \log_p(x) \rfloor + 1, & \text{if} \quad x = p^c - 1, \end{cases}$$

with $c \in \mathbb{N}$.

The proof is not difficult and is an easy calculus exercise, so we shall omit it here.

We now have the following

Corollary 2.9.

$$\ell(m, x+1) = \begin{cases} \ell(m,x), & \text{if} \quad x \neq p^c - 1 \quad \text{and} \quad p \nmid m; \\ p\,\ell(m,x), & \text{if} \quad x = p^c - 1 \quad \text{and} \quad p \mid m, \end{cases}$$

with $x, m \in \mathbb{N}$.

The proof of the above corollary comes from Definition 2.6 and Theorem 2.8.

From Lemma 2.1

$$\sum_{j=x}^{x+k-1} \binom{j}{x} = \binom{x+k}{x+1}.$$

The binomial cœfficient $\binom{x+k}{x+1}$ is well defined for $x \in \mathbb{N}$. Nevertheless, it was remarked in [5] that we can extend possibly the definition of $\binom{n}{x}$ (where it is implied that $0 \leq x \leq n$) to negative n.

Below we discuss a few general results and give a few general comments.

Using Pascal's rule, we can observe that

$$\binom{x+k}{x+1} + \binom{x+k}{x} = \binom{x+k+1}{x+1}.$$

Since $\binom{x+k}{x} \equiv \binom{x}{x} \equiv 1 \pmod m$, we obtain

$$\binom{x+k}{x+1} + 1 \equiv \binom{x+k+1}{x+1} \pmod m. \tag{2.1}$$

If $x \neq p^c - 1$ and $p|m$, then from the corollary above, we have $k = \ell(m,x) = \ell(m,x+1)$. So

$$\binom{x+k+1}{x+1} \equiv \binom{x+1}{x+1} \equiv 1 \pmod m,$$

and hence

$$\binom{x+\ell(m,x)}{x+1} \equiv 0 \pmod m.$$

If $x = p^c - 1$ and $p|m$, then from the corollary above, we have $pk = p\,\ell(m,x) = \ell(m,x+1)$. Now from Theorem 1.5 we have for $x = p^c - 1$ and $m = p$ a prime with $c \in \mathbb{N}$,

$$\binom{x+k+1}{x+1} = \binom{p^c + \ell(p,p^c-1)}{p^c} \equiv \left\lfloor \frac{p^c + \ell(p,p^c-1)}{p^c} \right\rfloor \equiv \left\lfloor \frac{\ell(p,p^c-1)}{p^c} \right\rfloor + 1 \pmod p. \tag{2.2}$$

From (2.1) and (2.2) with $x = p^c - 1$, $k = \ell(m,x)$ and $m = p$ a prime we have

$$\binom{p^c - 1 + \ell(p,p^c-1)}{p^c} \equiv \left\lfloor \frac{\ell(p,p^c-1)}{p^c} \right\rfloor \pmod p.$$

We have $\ell(p,p^c) = p^{c+1} = p\,\ell(p,p^c-1)$, so it follows that $\ell(p,p^c-1) = p^c$ and hence $\lfloor \frac{\ell(p,p^c-1)}{p^c} \rfloor = 1$ for $c \in \mathbb{N}$. Thus

$$\binom{2p^c-1}{p^c} \equiv 1 \pmod p.$$

Thus, we now have the following result.

Theorem 2.10. *For a prime p and a natural number c, we have*

$$\binom{2p^c-1}{p^c} \equiv 1 \pmod p.$$

In general, if $x = p^c - 1$ and $p|m$, then since $\lfloor \log_p(p^c) \rfloor = c$, and from Corollary 2.9 we have

$$\ell(m,p^c-1) = \frac{\ell(m,p^c)}{p} = mp^{c-1} \prod_{i \in [\![1,k]\!]\,|\,p_i \neq p} p_i^{\lfloor \log_{p_i}(p^c) \rfloor}.$$

If $b_i = \operatorname{ord}_{p_i}(m) = \lfloor \log_{p_i}(p^c) \rfloor$ for $i \in [\![1,k]\!]\,|\,p_i \neq p$ and $b = \operatorname{ord}_p(m)$, we have

$$\ell(m,p^c-1) = \frac{\ell(m,p^c)}{p} = m\,p^{c-1} \prod_{i \in [\![1,k]\!]\,|\,p_i \neq p} p_i^{b_i} = m\,p^{c-1} \times \frac{m}{p^b}.$$

So, we deduce that

$$\ell(m,p^c-1) = m^2\,p^{c-b-1} = \frac{m^2}{p^{b+1}} p^c,$$

and

$$\ell(m, p^c) = m^2\, p^{c-b}.$$

In particular, when $m = p^b$ we have $\ell(m = p^b, p^c - 1) = p^{b+c-1}$. And, we get $\ell(m = p, p^c - 1) = p^c$. If $c \geq \ \mathrm{ord}_p(m) + 1$, then $\ell(m, p^c - 1)$ is divisible by m^2. If $b = c$, then $\ell(m = p^c, p^c - 1) = \frac{m^2}{p}$.

Now from (2.2) and the preceeding paragraph we have,

$$\binom{x+k+1}{x+1} = \binom{p^c + \ell(m, p^c - 1)}{p^c} \equiv \left\lfloor \frac{p^c + \ell(m, p^c - 1)}{p^c} \right\rfloor \equiv \left\lfloor \frac{m^2}{p^{b+1}} \right\rfloor + 1 \,(\mathrm{mod}\, p).$$

From (2.1) with $x = p^c - 1$, $k = \ell(m, x)$, and also from the fact that $d \equiv e \,(\mathrm{mod}\, m)$ and $p|m$ implies that $d \equiv e \,(\mathrm{mod}\, p)$ (the converse is not always true), we have for $p|m$,

$$\binom{p^c - 1 + m_c^2\, p^{c-b-1}}{p^c} \equiv \left\lfloor \frac{m_c^2}{p^{b+1}} \right\rfloor \,(\mathrm{mod}\, p).$$

Remarks on Theorem 1.4

In the proof of Theorem 1.4, the authors in [5] first proved that a period of a sequence (a_n) such that $a_n \equiv \binom{n}{m} \,(\mathrm{mod}\, m)$ with $m = \prod_{i=1}^k p_i^{b_i}$, should be a multiple of the number $\ell(m) = m \prod_{i=1}^k p_i^{\lfloor \log_{p_i}(m) \rfloor}$. Afterwards, it is proved that $\ell(m)$ represents really the minimal period of such a sequence namely for every natural number n,

$$\binom{n + \ell(m)}{m} \equiv \binom{n}{m} \,(\mathrm{mod}\, m).$$

For that, the authors notice that it suffices to prove

$$\frac{\prod_{i=0}^{m-1}(n-i)}{\prod_{j=1}^k p_j^{\vartheta_{p_j}(m)}} \equiv \frac{\prod_{i=0}^{m-1}(n + \ell(m) - i)}{\prod_{j=1}^k p_j^{\vartheta_{p_j}(m)}} \,(\mathrm{mod}\, m),$$

where $\vartheta_{p_j}(m)$ is the p_j-adic ordinal of $m!$ defined as

$$\vartheta_{p_j}(m) = \ \mathrm{ord}_{p_j}(m!) = \sum_{l \geq 1} \left\lfloor \frac{m}{p_j^l} \right\rfloor = \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j^l} \right\rfloor. \tag{2.3}$$

Thus to prove Theorem 1.4 it is sufficient to show

$$\frac{\prod_{i=1}^{m}(n-i+1)}{\prod_{j=1}^k p_j^{\vartheta_{p_j}(m)}} \equiv \frac{\prod_{i=1}^{m}(n + \ell(m) - i + 1)}{\prod_{j=1}^k p_j^{\vartheta_{p_j}(m)}} \,(\mathrm{mod}\, m).$$

Then, the authors observe that among the numbers $n, n-1, \ldots, n-m+1$, there are at least $\lfloor \frac{m}{p^l} \rfloor$ that are divisible by p^l for every positive integer l and any prime p which appears in the prime factorization of m. In particular, if p divides m, we can notice that among the numbers $n, n-1, \ldots, n-m+1$ (which represents m consecutive numbers), there are exactly $\lfloor \frac{m}{p} \rfloor = \frac{m}{p}$ that are divisible by p for any prime p which appears in the prime factorization of m.

In the following, we define natural numbers $c_j(i)$ with $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$ by

$$\vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i)$$

such that the $c_j(i)$'s are functions of $\operatorname{ord}_{p_j}(m - i + 1)$ namely $c_j(i) = (\operatorname{ord}_{p_j}(n - i + 1))$ and $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$. Also $c_j(i) = 0$ if $\operatorname{ord}_{p_j}(m - i + 1) = 0$.

We now state and prove the following result.

Theorem 2.11. *If*

$$\max\left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \leq \lfloor \log_{p_j}(m) \rfloor$$

then

$$\vartheta_{p_j}(m) \leq \lfloor \frac{m}{p_j} \rfloor \lfloor \log_{p_j}(m) \rfloor.$$

(In general, the converse is not always true.) Therefore, a necessary but not sufficient condition in order to satisfy the inequality $\vartheta_{p_j}(m) \leq \lfloor \frac{m}{p_j} \rfloor \lfloor \log_{p_j}(m) \rfloor$, *is*

$$c_j(i) \leq \lfloor \log_{p_j}(m) \rfloor, \;\; \forall i \in [\![1, m]\!]$$

with $j = 1, 2, \ldots, k$.

Proof. The proof is immediate from (2.3) by noticing the following,

$$\operatorname{ord}_{p_j}(m!) = \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j^l} \right\rfloor \leq \sum_{l=1}^{\lfloor \log_{p_j}(m) \rfloor} \left\lfloor \frac{m}{p_j} \right\rfloor = \left\lfloor \frac{m}{p_j^l} \right\rfloor \lfloor \log_{p_j}(m) \rfloor.$$

□

We can notice that this choice is not unique. But, we can observe that all the choices for the $c_j(i)$'s are equivalent in the sense that the equality $\vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i)$ should hold, meaning that we can come back to a decomposition of the value of $\vartheta_{p_j}(m)$ into sum of positive numbers like the $c_j(i)$'s for which $c_j(i) \leq \lfloor \log_{p_j}(m) \rfloor$ with $i = 1, 2, \ldots, m$. It turns out that this choice is suitable in order to prove that $\ell(m)$ is the minimal period of sequences (a_n) such that $a_n \equiv \binom{n}{m} \pmod m$ with $m = \prod_{i=1}^{k} p_i^{b_i}$ (with at least one non-zero b_i).

Remark 2.12. *We have obviously*

$$\max\left\{ c_j(i) \;\middle|\; \vartheta_{p_j}(m) = \sum_{i=1}^{m} c_j(i) \right\} \geq 1.$$

The above discussion gives us a motivation to study the coefficients $c_j(i)$'s. We hope to address a few issues related to them and establish some interesting results in a forthcoming paper.

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On p-adic analogues of certain Ramanujan type formulas for $\frac{1}{\pi}$: a brief survey

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Abstract. Van Hamme developed p-adic analogues of certain series representation of $\frac{1}{\pi}$, known as Ramanujan-type congruences. We shall discuss briefly different methods developed for the proofs of p-adic analogues of Ramanujan-type congruences. The aim of this article is to encourage interested readers to go through the works described briefly here.

Keywords. Ramanujan-type Supercongruences.

1 Introduction

In 1914, Ramanujan [13] listed 17 infinite series representations of $\frac{1}{\pi}$, proved by J. Borwein and P. Borwein [1], of the form

$$\sum_{k=0}^{\infty} A(k)x^k = \frac{\delta}{\pi},$$

which gained popularity when they were used to calculate digits of π. In [5], Van Hamme developed p-adic analogs of these series, known as Ramanujan type supercongruences, which related truncated hypergeometric series to the values of p-adic gamma functions. We list them in Table 1 of the next section, and then review developments of their proofs. Note that $S(m)$ denotes the left column corresponding sum truncated at $k = m$ and in the last supercongruence (M.2) $\beta(n)$ denotes the nth Fourier co-efficient of the eta-product

$$f(z) := \eta(2z)^4\eta(4z)^4 = q\prod_{n\geq 1}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n\geq 1}\beta(n)q^n,$$

where $q = e^{2\pi iz}$.

2 Method of proofs of Van Hamme conjectures

- **[(C.2), (H.2), (I.2)]**: Providing p-adic analogs for the Ramanujan type supercongruences, Van Hamme gave proofs of (C.2), (H.2), and (I.2) using various methods. For example, he used a sequence of orthogonal polynomials $p_k(x)$ which satisfy a certain recurrence relation, and then analyse these polynomials to yield (C.2).

- **[(A.2)]**: McCarthy and Osburn [9] proved (A.2) using Gaussian hypergeometric series, properties of p-adic Gamma functions, and certain strange combinatorial identities. They used the following result due to Osburn and Schneider [11] which states that for an odd prime p and integer $n \geq 2$,

$$-p^n {}_{n+1}F_n(\lambda) \equiv (-1)^{n+1} \left(\frac{-1}{p}\right)^{n+1} [p^2 X(p,\lambda,n) + pY(p,\lambda,n) + Z(p,\lambda,n)] \pmod{p^3},$$

  where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol and $X(p,\lambda,n), Y(p,\lambda,n), Z(p,\lambda,n)$ are quantities involving generalized harmonic sums $H_n^{(i)} := \sum_{j=1}^{n} \frac{1}{j^i}$. For $p \equiv 1 \pmod 4$, Swisher [14] improved (A.2), and proved that it holds in modulo p^5 for $p > 5$.

- **[(B.2)]**: Three different methods have been used to prove (B.2). Mortenson [10] proved using a technical evaluation of a quotient of Gamma functions, Zudilin [16] used the WZ-method designed by Wilf and Zeilberger [15], and Long [8] used hypergeometric series identities and evaluations.

- **[(J.2)]**: Long [8] also used a similar but more general method to prove (J.2). Motivated by the techniques of McCarthy and Osburn [9], Mortenson [10], and Zudilin [16], she utilized suitable hypergeometric series identities to obtain (J.2). She further gave a more general result and proved several other supercongruences related to special valuations of truncated hypergeometric series.

  Theorem 2.1. *Let $p > 3$ be a prime and r be a positive integer. Then*

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \left(\frac{(\frac{1}{2})}{k!}\right)^4 \equiv p^r \pmod{p^{3+r}}.$$

- **[(D.2), (H.2)]**: Recently, Long and Ramakrishna [7] have proved (D.2). They have used relations between classical hypergeometric series and p-adic Gamma functions to strengthen and extend (D.2) and (H.2) to additional primes. In fact, they extended (D.2) to modulo p^6, and proved that (H.2) holds modulo p^3 when $p \equiv 1 \pmod 4$.

- **[(E.2), (F.2), (G.2), (L.2)]**: The framework of Long [8] used in the proof of (B.2) is utilized by many mathematicians to find similar congruences for truncated hypergeometric series in terms of p-adic Gamma functions. Motivated by the work of Long [8], Swisher [14] used identities of classical hypergeometric series to prove (E.2), (F.2), (G.2), and (L.2), and extended some of them to additional primes. She further improved many results of

Ramanujan Series	Conjectures of van Hamme
(A.1) $\sum_{k=0}^{\infty}(4k+1)(-1)^k\frac{(\frac{1}{2})_k^5}{k!^5}=\frac{2}{\Gamma(\frac{3}{4})^4}$	(A.2) $S\left(\frac{p-1}{2}\right)\equiv\begin{cases}\frac{-p}{\Gamma_p\left(\frac{3}{4}\right)^4} & \text{if } p\equiv 1 \pmod 4\\ 0 & \text{if } p\equiv 3 \pmod 4\end{cases}\pmod{p^3}$
(B.1) $\sum_{k=0}^{\infty}(4k+1)(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2}{\pi}=\frac{2}{\Gamma(\frac{1}{2})^2}$	(B.2) $S\left(\frac{p-1}{2}\right)\equiv\frac{-p}{\Gamma_p\left(\frac{1}{2}\right)^2}\pmod{p^3}$, $p\neq 2$
(C.1) $\sum_{k=0}^{\infty}(4k+1)\frac{(\frac{1}{2})_k^4}{k!^4}=\infty$	(C.2) $S\left(\frac{p-1}{2}\right)\equiv p \pmod{p^3}$, $p\neq 2$
(D.1) $\sum_{k=0}^{\infty}(6k+1)\frac{(\frac{1}{3})_k^6}{k!^6}$ =1.01226...	(D.2) $S\left(\frac{p-1}{3}\right)\equiv -p\Gamma_p\left(\frac{1}{3}\right)^9\pmod{p^4}$, if $p\equiv 1 \pmod 6$
(E.1) $\sum_{k=0}^{\infty}(6k+1)(-1)^k\frac{(\frac{1}{3})_k^3}{k!^3}=\frac{3\sqrt{3}}{2\pi}=\frac{3}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}$	(E.2) $S\left(\frac{p-1}{3}\right)\equiv p \pmod{p^3}$, if $p\equiv 1 \pmod 6$
(F.1) $\sum_{k=0}^{\infty}(8k+1)(-1)^k\frac{(\frac{1}{4})_k^3}{k!^3}=\frac{2\sqrt{2}}{\pi}=\frac{4}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$	(F.2) $S\left(\frac{p-1}{4}\right)\equiv\frac{-p}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}\pmod{p^3}$, if $p\equiv 1 \pmod 4$
(G.1) $\sum_{k=0}^{\infty}(8k+1)\frac{(\frac{1}{4})_k^4}{k!^4}=\frac{2\sqrt{2}}{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)^2}$	(G.2) $S\left(\frac{p-1}{4}\right)\equiv p\frac{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{4}\right)}{\Gamma_p\left(\frac{3}{4}\right)}\pmod{p^3}$, if $p\equiv 1 \pmod 4$
(H.1) $\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{\pi}{\Gamma\left(\frac{3}{4}\right)^4}$	(H.2) $S\left(\frac{p-1}{2}\right)\equiv\begin{cases}-\Gamma_p\left(\frac{1}{4}\right)^4 & \text{if } p\equiv 1 \pmod 4\\ 0 & \text{if } p\equiv 3 \pmod 4\end{cases}\pmod{p^2}$
(I.1) $\sum_{k=0}^{\infty}\frac{1}{k+1}\frac{(\frac{1}{2})_k^2}{k!^2}=\frac{4}{\pi}=\frac{4}{\Gamma\left(\frac{1}{2}\right)^2}$	(I.2) $S\left(\frac{p-1}{2}\right)\equiv 2p^2 \pmod{p^3}$, $p\neq 2$
(J.1) $\sum_{k=0}^{\infty}\frac{6k+1}{4^k}\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{4}{\pi}=\frac{4}{\Gamma\left(\frac{1}{2}\right)^2}$	(J.2) $S\left(\frac{p-1}{2}\right)\equiv\frac{-p}{\Gamma_p\left(\frac{1}{2}\right)^2}\pmod{p^4}$, $p\neq 2,3$
(K.1) $\sum_{k=0}^{\infty}\frac{42k+5}{64^k}\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{16}{\pi}=\frac{16}{\Gamma\left(\frac{1}{2}\right)^2}$	(K.2) $S\left(\frac{p-1}{2}\right)\equiv\frac{-5p}{\Gamma_p\left(\frac{1}{2}\right)^2}\pmod{p^4}$, $p\neq 2$
(L.1) $\sum_{k=0}^{\infty}\frac{6k+1}{8^k}(-1)^k\frac{(\frac{1}{2})_k^3}{k!^3}=\frac{2\sqrt{2}}{\pi}=\frac{4}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$	(L.2) $S\left(\frac{p-1}{2}\right)\equiv\frac{-p}{\Gamma_p\left(\frac{1}{4}\right)\Gamma_p\left(\frac{3}{4}\right)}\pmod{p^3}$, $p\neq 2$
(M.1) $\sum_{k=0}^{\infty}\frac{(\frac{1}{2})_k^4}{k!^3}$: unknown	(M.2) $S\left(\frac{p-1}{2}\right)\equiv\beta(p) \pmod{p^3}$, $p\neq 2$

Table 2.1: The Van Hamme Conjectures

Van Hamme to modulo of higher prime powers. Using similar methods, He [3, 4] has also independently verified the results for cases (E.2), (F.2), and (G.2). In [14], Swisher also listed a number of Van Hamme-type congruence conjectures based on computational evidence computed using Sage, some of which have been deduced by He [2].

Theorem 2.2. *Let n be a positive integer and p be an odd prime with $p\equiv -1 \bmod n$. Then*

$$ {}_4F_3\left[\begin{matrix}1+\frac{1}{2n}, & \frac{1}{n}, & \frac{1}{n}, & \frac{1}{n}\\ \frac{1}{2n}, & 1, & 1 & \end{matrix}\middle| 1\right]_{\frac{p^2-1}{n}}\equiv p^2 \pmod{p^4}. $$

- **[(K.2)]**: A new approach based on classical congruences and a WZ pair due to Guillera was used by Osburn and Zudilin [12] in order to prove the congruence (K.2) completing proof of all Van Hamme conjectures.
- **[(M.2)]**: Interestingly the conjecture (M.2) is related to the Apéry number supercongruence. Using the fact that $\beta(p)$ in (M.2) is related to a modular Calabi-Yau threefold, Kilbourn [6] proved (M.2).

3 Conclusion

As already described, the proofs of Van Hamme conjectures motivated mathematicians to explore the relations similar to Van Hamme in higher powers of primes than expected by him. Several other new results have also been found exploring connections of new hypergeometric series to p-adic gamma functions.

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A brief study on approximate controllability via semigroup theory

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Abstract. This article is a brief survey on approximate controllability of a semilinear control system in abstract spaces with the help of semigroup theory of bounded linear transformation. Amongst the various approaches to the study of the controllability of non linear systems, we mainly focus on the fixed point method. As an application, we discuss the work done by Mahmudov[3] on controllability.

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Keywords. controllability, semigroup, fixed point theorem, semilinear system.

1 Introduction

The concept of controllability was introduced by an American mathematician and electrical engineer R. Kalman in 1960. Kalman is popularly known for his greatest invention, Kalman filter, a series of algorithms basically a branch of mathematical formulations that precisely gives an idea to estimate the state of a system. The fundamental idea of Kalman filter is to extract less noisy data from a given set of noisy data. It has a numerous application in the field of technology. Kalman filter has a remarkable influence on sending the first human spaceflight from the Earth to the Moon. For this work Kalman has received the prestigious National Medal of Science from the U.S. president Barack Obama in 2009.

Moving to controllability, it is one of the broad concept in modern control system. It is a process of driving a dynamical system to a particular state with the help of control input. If a system is not controllable that means no input will be able to control that system. The general concept of a system being controllable is that if we have system and want to make movements from one point to another point within the system by taking help of input or outside source then our system must be controllable. Depending upon our system, there are various approaches to study of controllability. For example, the study of controllability for infinite dimensional system, we may require more assumption compared to finite dimensional system.

Basically our aim is to study about approximate controllability of a semilinear system. Amongst the various approaches to the study of controllability of nonlinear systems, we use the fixed point method for our purpose. In the fixed point method, the problem of controllability is transformed to a fixed point problem for a nonlinear operator in a function space and then try to find out conditions and assumptions needed to solve the problem.

We organise our article as, in the section 2, we state basic results from semigroup theory. In the section 3, we briefly discuss controllability of finite dimensional space and infinite dimensional space in both linear and nonlinear case. In the section 4, first we consider a semilinear system in Banach Space then try to prove approximate controllability for that system using results from previous sections.

2 Semigroup

Now we state some well known results from semigroup theory that will be required in our main discussion of the article.

Semigroups of Bounded Linear Operators

Definition 2.1. *A family of operators $\{T(t)\}_{t\geq 0}$, $T(t) : X \to X$ is said to be a semigroup of bounded linear operators on a complex Banach Space (BS) X if*

(i) $T(t) : X \to X$ *is a bounded linear operator, i.e.* $\|T(t)\|_X < \infty$.

(ii) $T(t+s)x = T(t)T(s)x$, *for each* $x \in X$ *and* $t, s \geq 0$.

(iii) $T(0) = I$, *where* I *is the identity operator on* X.

Example 2.2. *Consider,* $X = B \cup C(\mathbb{R})$ *with* $\|\cdot\|_\infty$ *norm.*

Clearly, $(X, \|\cdot\|_\infty)$ *is a BS.*

Define,

$$T(t) : X \to X, \text{ as}$$
$$T(t)f(x) = f(x+t), \ t \geq 0, x \in \mathbb{R}.$$

Then $T(t)$ *defined as above on* X *is a semigroup.*

Example 2.3. *Consider,* $X = B \cup C(\mathbb{R})$ *with* $\|\cdot\|_\infty$ *norm.*

Define,

$$T(t) : X \to X \text{ for a fixed } n \in \mathbb{N} \text{ as}$$
$$T(t)f(x) = e^{-nt} f(x+t), \ t \geq 0, x \in \mathbb{R}.$$

Then $T(t)$ *is a semigroup on* X.

Definition 2.4. *A semigroup $T(t)$ $(0 \leq t < \infty)$ of bounded linear operators on a BS X is said to be a strongly continuous semigroup(abbreviated C_0 semigroup) of bounded linear operators if*

$$\lim_{t \downarrow 0} T(t)x = x, \text{ for each } x \in X.$$

Definition 2.5. *Let $\{T(t)\}_{t \geq 0}$ be a semigroup of bounded linear operators on X. The infinitesimal generator $A : D(A) \subset X \to X$ of $T(t)$ is defined by*

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}, \ x \in D(A),$$

where $D(A)$ is the domain of A and it is defined as

$$D(A) = \Big\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \quad exists \Big\}.$$

Example 2.6. *We will find the infinitesimal generator of the semigroup $T(t)$ defined in the Example 2.2. Let A be the infinitesimal generator of $T(t)$. Then*

$$\begin{aligned} A(f(x)) &= \lim_{t \downarrow 0} \frac{T(t)f(x) - f(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{f(x+t) - f(x)}{t} = \frac{d^+}{dt} f(x). \end{aligned}$$

Hence,

$$D(A) = \{ f \in X | \ f \text{ is absolutely continuous and } f' \in X \}.$$

Remark 2.7. *The infinitesimal generator A of a semigroup $T(t)$ on X is a linear operator on X.*

Example 2.8. *Let A be a bounded linear operator on a BS X. Then for each $t \geq 0$, $T(t) = e^{tA}$ is a C_0 semigroup on X.*

Theorem 2.9. *[1] A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 2.10. *[1] Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If $x \in D(A)$ then $T(t)x \in D(A)$ and*

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax.$$

Theorem 2.11. *[1] Let $T(t)$ and $S(t)$ be two C_0 semigroups of bounded linear operators with infinitesimal generators A and B respectively. If $A = B$ then $T(t) = S(t)$ for $t \geq 0$.*

Corollary 2.12. *If A is the infinitesimal generator of a C_0 semigropup, then A is a closed linear operator on $D(A)$ and $\overline{D(A)} = X$.*

Theorem 2.13. *[1] Let $T(t)$ be a C_0 semigroup. Then there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for } 0 \leq t \leq \infty.$$

Corollary 2.14. *If $T(t)$ is a C_0 semigroup then for every $x \in X$, $t \mapsto T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Definition 2.15. *A C_0 semigroup $T(t)$ on X is said to be a uniformly bounded semigroup if $\exists$ $M > 0$ such that*

$$\|T(t)\| \leq M, \ \forall t \geq 0.$$

Definition 2.16. *A C_0 semigroup $T(t)$ on X is said to be a semigroup of contraction if*

$$\|T(t)\| \leq 1, \ \forall \ t \geq 0.$$

Definition 2.17. *The resolvent set $\rho(A)$ for an operator A from X to X is defined as*

$$\rho(A) = \{\lambda \in \mathbb{C} | A - \lambda I : D(A) \subset X \to X \text{ is bijective}\}.$$

And the resolvent operator $R(\lambda : A)$ is defined by

$$R(\lambda : A) = (A - \lambda I)^{-1}.$$

Theorem 2.18. *[1] A linear (unbounded) operator A is the infinitesimal generator of a C_0 semigroup of contractions $T(t)$ on X iff*

(i) A is closed and $\overline{D(A)} = X$.

(ii) The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and satisfies $\|R(\lambda : A)\| \leq \dfrac{1}{\lambda}$, for every $\lambda > 0$.

Example 2.19. *Let X be a separable Hilbert Space(abbreviated HS) with an orthonormal basis $\{v_n : n \geq 1\}$. Consider a sequence of real numbers $\{\lambda_n : n \geq 1\}$ such that sup $\{\lambda_n : n \geq 1\} < \infty$. Define an operator A on X by*

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n, \ \ x \in D(A),$$

with

$$D(A) = \Big\{ x \in X : \sum_{n=1}^{\infty} |\lambda_n \langle x, v_n \rangle|^2 < \infty \Big\}.$$

Then A generates a C_0 semigroup on X.

Semigroups of Compact Operators

Definition 2.20. *A C_0 semigroup $T(t)$ is called compact for $t > t_0$ if for every $t > t_0$, $T(t)$ is a compact operator. In general $T(t)$ is called compact if it is compact for $t > 0$.*

Theorem 2.21. *[1] Let $T(t)$ be a C_0 semigroup on X. If $T(t)$ is compact for $t > t_0$ then $T(t)$ is continuous in the uniform operator topology for $t > t_0$.*

Theorem 2.22. *[1] Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. Then $T(t)$ is a compact semigroup iff $T(t)$ is continuous in the uniform opertor topology for $t > 0$ and $R(\lambda : A)$ is compact for $\lambda \in \rho(A)$.*

Corollary 2.23. *Let A be the infinitesimal generator of a C_0 semigroup $T(t)$ on X. If $R(\lambda : A)$ is compact for some $\lambda \in \rho(A)$ and $T(t)$ is continuous in the uniform operator topology for $t > t_0$ then $T(t)$ is compact for $t > t_0$.*

Corollary 2.24. *Let $T(t)$ be a uniformly continuous semigroup. Then $T(t)$ is a compact semigroup iff $R(\lambda : A)$ is compact for every $\lambda \in \rho(A)$.*

Example 2.25. *Any linear bounded functional on a HS is a compact operator.*

Remark 2.26. *Adjoint of a compact operator is again a compact operator.*

Example 2.27. *Let X be a separable Hilbert Space with an orthonormal basis $\{v_n : n \geq 1\}$. Consider a sequence nonnegative real numbers $\{\lambda_n : n \geq 1\}$ such that sup $\{\lambda_n : n \geq 1\} < \infty$. Define an operator A on X by*

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n, \quad x \in X.$$

Then A is a compact operator on X.

Example 2.28. *The operator $A := \dfrac{d^2}{dx^2}$ with $D(A) := \{f \in C^2[0,1] : f'(0) = 0 = f'(1)\}$ generates a compact semigroup $T(t)$ on $X := C[0,1]$.*

3 Controllability

The problem of controllability is a mathematical formulation of the following situation. Consider a system which evolves with respect to(abbreviated w.r.t.) time

$$\dot{x} = f(t, x, u)$$

where x is a description of the state of the system, $\dot{x}$ represents derivative of x w.r.t. time t and u is the control which can be chosen in a particular range.The standard problem of controllability can be viewed as:
Given a finite time $T > 0$, an initial state x_0 and the final state x_1, is it possible to find a control function u(depending on the time) such that solution of the system, starting from x_0 and provided with this function u reaches the state x_1 at time T?

Depending upon the system the controllability problem can be viewed in two different ways. If the state of the system can be described by a finite numbers of degrees of freedom, we call the problem finite dimensional. On the other hand if the state of the system can not be described by finite numbers of degrees of freedom then we term the problem as infinite dimensional. In infinite dimensional problem we use various results from semigroup theory. First we will discuss about finite dimensional and after that we will discuss about infinite dimensional problem.

Finite Dimensional System

Linear System

Let us consider the linear system,

$$\dot{x} = Ax + Bu,$$

where $x \in \mathbb{R}^n$ is a state vector ; $A \in \mathbb{M}_{n\times n}$.

And $u \in \mathbb{R}^m$ is a input vector ; $B \in \mathbb{M}_{n\times m}$.

Definition 3.1. *The pair (A, B) is said to be controllable if given a duration $T > 0$ and two arbitrary points $x_0, x_1 \in \mathbb{R}^n$, there exists a piecewise continuous function $\overline{u} : [0, T] \to \mathbb{R}^m$ such that the solution of the given system $\overline{x}(t)$ generated by $\overline{u}$ with $\overline{x}(0) = x_0$ satisfies*

$$\overline{x}(T) = x_1.$$

In other words

$$x_1 = e^{AT}x_0 + \int_0^T e^{A(T-t)}B\overline{u}(t)dt.$$

Theorem 3.2. *A necessary and sufficient condition for the pair (A, B) to be controllable is*

$$rank\ C = \ rank\big(B|AB|\cdots|A^{n-1}B\big) = n.$$

The matrix $C = \big(B|AB|\cdots|A^{n-1}B\big)$ is called Kalman's controllability matrix of size $(n \times nm)$.

Example 3.3. *The system $\dot{x_1} = x_2$, $\dot{x_2} = u$ is controllable.*
This system can be written in the standard form as

$$\begin{pmatrix}\dot{x_1}\\ \dot{x_2}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u.$$

Here,

$$A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, \quad B = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

Hence,

$$C = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad and \quad rank(C) = 2.$$

This implies that the given system is controllable.

Example 3.4. *The system $\dot{x_1} = u$, $\dot{x_2} = u$ is not controllable.*

Nonlinear System

Consider the non linear system

$$\dot{x} = f(x, u) \tag{3.1}$$

with $x \in X$, n dimensional manifold and $u \in \mathbb{R}^m$ is a input vector.

We will convert the nonlinear system (3.1) to a linear form by introducing *equilibrium point.*

Let $(\overline{x}, \overline{u})$ be a equilibrium point of (3.1). Then the corresponding linear form of (3.1) about the equilibrium point $(\overline{x}, \overline{u})$ is given by

$$\dot{\xi} = A\xi + B\nu, \tag{3.2}$$

where, $A = \dfrac{\partial f}{\partial x}(\overline{x}, \overline{u})$ and $B = \dfrac{\partial f}{\partial u}(\overline{x}, \overline{u})$.

Definition 3.5. *The non linear system (3.1) is first order controllable around a equilibrium point $(\overline{x}, \overline{u})$ if its linear system (3.2) is controllable at $(\overline{x}, \overline{u})$ i.e iff*

$$rank\big(B|AB|\ldots|A^{n-1}B\big) = n.$$

Infinite Dimensional System

In the case of infinite dimensional systems two different types of controllability can occur. These are exact and approximate controllability. Exact controllability enables to steer the system to arbitrary final state while approximate controllability means that the system can be steered to arbitrary small neighborhood of the final state. It is obvious that exact controllability is essentially stronger notion than approximate controllability. In other words, exact controllability always implies approximate controllability. The converse statement is generally false. However, in the case of infinite dimensional systems exact controllability appears rather exceptionally.

Consider the following linear system in a $finite$ time interval $I = [0, T]$,

$$\left.\begin{aligned} z\prime(t) &= Az(t) + Bu(t),\ t \geq 0, \\ z(0) &= z_0, \end{aligned}\right\} \tag{3.3}$$

where A is the infinitesimal generator of a C_0 semigroup $S(t)$ on a HS Z. $B : U \to Z$ is a *bounded* linear operator, where U is a HS. $u \in L^2(I, U)$ is considered as the control function. We denote the system (3.3) as (A, B) for further use.

A mild solution $z(\cdot)$ of the linear system (3.3) is given by the solution of the following integral equation,

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds.$$

Definition 3.6. *Some important definitions are listed below by considering the linear system (3.3).*

(i) *Controllability Map* : *The controllability map of the system (3.3) on I is the bounded linear map $\Gamma^T : L^2(I,U) \to Z$ defined by*

$$\Gamma^T u := \int_0^T S(T-s)Bu(s)ds.$$

(ii) *Reachability Set* : *Let $z(T, z_0, u)$ be the state value at time T corresponding to the control $u(\cdot)$ and initial value z_0. Define,*

$$R(T, z_0, u) = \{z(T, z_0, u) : \ u(\cdot) \ \text{ is a control, } u \in L^2(I,U)\}.$$

The set $R(T, z_0, u)$ is known as reachability set.

(iii) *Exact Controllability* : *The system (3.3) is exactly controllable on I if all points in Z can be reached from origin at time T, ($T > 0$),*

$$i.e. \ R(T, z_0, u) = Z.$$

(iv) *Approximate Controllability* : *The system (3.3) is approximately controllable on I if given an arbitrary $\varepsilon > 0$ it is possible to steer from the origin to within a distance ε from all points in the state space at time T.*

$$i.e. \ \overline{R(T, z_0, u)} = Z.$$

(v) *Controllability Gramian* : *The controllability gramian of the system (3.3) on I is defined by*

$$\Gamma_0^T := \Gamma^T(\Gamma^T)^*.$$

Lemma 3.7. *The controllability map and controllability gramian as defined in the previous definition satisfy the following :*

(i) $\Gamma^T \in L\big(L^2(I:U), Z\big)$ *and* $\Gamma^t \in L\Big(L^2(I:U), L^2(I:Z)\Big)$ *for* $0 \le t \le T$.

(ii) $(\Gamma^T)^* z(s) = B^* S^*(T-s)z$ *on* I.

(iii) $\Gamma_0^T \in L(Z)$ *and* $\Gamma_0^T z = \int_0^T S(t)BB^*S^*(t)zdt$ *for* $z \in Z$.

Lemma 3.8. *The system (3.3) i.e the system (A, B) is exactly(approximately) controllable on I if and only if $(\mu I + A, B)$ is for any $\mu \in \mathbb{C}$.*

Theorem 3.9. *[2] The system (3.3) is exactly controllable on I if and only if any one of the following conditions hold for some $\gamma > 0$ and all $z \in Z$:*

(i) $\langle \Gamma_0^T z, z\rangle \ge \gamma \|z\|_Z^2$,

(ii) $\|(\Gamma^T)^* z\|_2^2 := \int_0^T \|(\Gamma^T)^* z(t)\|_U^2 dt \ge \gamma \|z\|_Z^2$,

(iii) $\int_0^T \|B^* S^*(t)z\|_U^2 dt \ge \gamma \|z\|_Z^2$,

(iv) ker $(\Gamma^T)^* = 0$ *and ran* $(\Gamma^T)^*$ *is closed.*

Theorem 3.10. *[2] The system (3.3) is approximate controllable on* I *if and only if any one of the following conditions hold :*

(i) $\Gamma_0^T > 0$,

(ii) $ker(\Gamma^T)^* = 0$,

(iii) $B^*S^*(t)z = 0$ *on* $I \implies z = 0$.

4 Approximate Controllability of Semilinear Evolution Equations

In this section we state the work of [3]. Now we state some new properties of *positive operator* required to prove the main result of Mahmudov[3]. For this we consider Z to be a *reflexive BS*, $\Gamma : Z^* \to Z$ is a *symmetric operator* and $J : Z \to Z^*$ be the duality mapping given by the following relations

$$\|J(z)\| = \|z\|, \langle J(z), z\rangle = \|z\|^2 \text{ for all } z \in Z.$$

Lemma 4.1. *[3] Let* Γ *be a non-negative symmetric operator. Then there exists a HS* H *and an operator* $A \in L(Z^*, H)$ *such that* $A^*A = \Gamma$ *and* $A(Z^*)$ *is dense in* H. *Furthermore* $A^*(H) \subset Z$ *and* $\overline{\Gamma(Z^*)} = \overline{A^*(H)}$.

Lemma 4.2. *[3] For every* $h \in Z$ *and* $\alpha > 0$ *the equation*

$$\alpha z_\alpha + \Gamma J(z_\alpha) = \alpha h \tag{4.1}$$

has a unique solution $z_\alpha = z_\alpha(h) = \alpha(\alpha I + \Gamma J)^{-1}(h)$ *and*

$$\|z_\alpha(h)\| = \|J(z_\alpha(h))\| \leq \|h\| \tag{4.2}$$

Theorem 4.3. *[3] Let* Γ *be a symmetric operator. Then the following three statements are equivalent :*

(i) Γ *is a positive operator.*

(ii) For all $h \in Z$, $J(z_\alpha(h))$ *converges to zero as* $\alpha \to 0^+$ *in the weak topology. where* $z_\alpha(h) = \alpha(\alpha I + \Gamma J)^{-1}(h)$ *is a solution of eq*^{n} (3.1).

(iii) For all $h \in Z$, $z_\alpha(h) = \alpha(\alpha I + \Gamma J)^{-1}(h)$ *converges to zero as* $\alpha \to 0^+$ *in the strong topology.*

Theorem 4.4. *[3] Let* Γ *be a positive symmetric operator and let* $h : Z \to Z$ *be a nonlinear operator. Assume* z_α *is a solution of the equation*

$$\alpha z_\alpha + \Gamma J(z_\alpha) = \alpha h(z_\alpha) \tag{4.3}$$

and

$$\|h(z_\alpha) - \bar{h}\| \to 0 \text{ as } \alpha \to 0^+.$$

Then there exists a subsequence of the sequence $\{z_\alpha\}$ *converging strongly to zero as* $\alpha \to 0^+$.

Consider a semilinear evolution equation on a separable reflexive BS X in the following form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(t, x(t), u(t)), \qquad t \in I = [0, T], \\ x(0) &= x_0, \end{aligned}\right\} \tag{4.4}$$

where $f : I \times X \times U \to X$ is a nonlinear operator. In this section we prove the approximate controllability of the system (4.4) under some assumptions.

We consider the following assumptions :

(A1) X is a reflexive BS and U is a separable HS.

(A2) The linear operator $A : D(A) \subset X \to X$ generates a compact semigroup $S(t)$, $t > 0$ on X.

(A3) The function $f : I \times X \times U \to X$ is continuous and there exist functions $\lambda_i(\cdot) \in L^1(I, \mathbb{R}^+)$ and $\phi_i(\cdot) \in L^1(X \times U, \mathbb{R}^+)$, $i = 1, 2, 3 \ldots$ such that

$$\|f(t, x, u)\| \leq \sum_{i=1}^{q} \lambda_i(t)\phi_i(x, u) \text{ for all } (t, x, u) \in I \times X \times U.$$

(A4) For all $\alpha > 0$ $limsup_{r\to\infty}\Big(r - \sum_{i=1}^{q} \frac{c_i}{\alpha} sup\{\phi_i(x, u) : \|(x, u)\| \leq r\}\Big) = \infty$.

(A5) For every $h \in X$, $z_\alpha(h) = \alpha(\alpha I + \Gamma_0^T J)^{-1}(h)$ converges to zero as $\alpha \to 0^+$ in strong topology. Where,

$L_0^T u := \int_0^T S(T - s)Bu(s)ds$,

$\Gamma_0^T := \int_0^T S(T - s)BB^*S^*(T - s)ds = L_0^T(L_0^T)^*$,

and $z_\alpha(h)$ is a solution of the equation

$$\alpha z_\alpha + \Gamma_0^T J(z_\alpha) = \alpha h,$$

where J is already defined.

(A6) The system (4.4) is approximate controllable if for all $\alpha > 0$ there exists a pair of continuous function $(x, u)(\cdot) \in C(I, X \times U)$ such that

$$\left.\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t - s)[Bu(s) + f(s, x(s), u(s))]ds, \\ u(t) &= B^*S^*(T - t)J\Big((\alpha I + \Gamma_0^T J)^{-1}p(x, u)\Big), \end{aligned}\right\} \tag{4.5}$$

where,

$$p(x, u) = x_T - S(T)x_0 - \int_0^T S(T - s)f(s, x(s), u(s))ds.$$

Theorem 4.5. *[3] Suppose that assumptions from* $(A1)$ *to* $(A4)$ *are satisfied. Then the operator* P_α*, defined on* $C(I, X \times U)$ *as*

$$P_\alpha(x, u) = (z, v), \tag{4.6}$$

where,

$$v(t)(= v_\alpha(t)) = B^* S^*(T-t) J((\alpha I + \Gamma_0^T J)^{-1} p(x, u)), \tag{4.7}$$

$$z(t)(= z_\alpha(t)) = S(t)x_0 + \int_0^t S(t-s)[Bv_\alpha(s) + f(s, x(s), u(s))]ds, \tag{4.8}$$

$$p(x, u) = x_T - S(T)x_0 - \int_0^T S(T-s) f(s, x(s), u(s))ds,$$

has a fixed point for $0 < \alpha \leq 1$.

Lemma 4.6. *[3] If*

$$p = x_T - S(T)x_0 - \int_0^T S(T-s) f(s)ds$$

with $f(\cdot) \in L^1(I, X)$ *and if* $u_\alpha(\cdot) \in L^2(I, X)$ *is a control defined by*

$$u_\alpha(t) = B^* S^*(T-t) J\Big((\alpha I + \Gamma_0^T J)^{-1} p\Big), \tag{4.9}$$

then

$$z(T; x_0, u_\alpha) - x_T = -\alpha(\alpha I + \Gamma_0^T J)^{-1} p, \tag{4.10}$$

and

$$z(t; x_0, u_\alpha) = S(t)x_0 + \int_0^t S(t-s) f(s)ds + \Gamma_0^t S^*(T-t) J\Big((\alpha I + \Gamma_0^T J)^{-1} p\Big), \tag{4.11}$$

where,

$$z(t; x_0, u) = S(t)x_0 + \int_0^t S(t-s)[Bu(s) + f(s)]ds. \tag{4.12}$$

Theorem 4.7. *[3] Let the function* $f : I \times X \times U \to X$ *be continuous and uniformly bounded i.e. there exists a constant* $L > 0$ *such that*

$$\|f(t, x, u)\| \leq L, \quad \textit{for all} \quad (t, x, u) \in I \times X \times U,$$

and the assumptions $(A1)$*,* $(A2)$ *and* $(A5)$ *are satisfied then the system (4.4) is approximate controllable.*

5 Conclusion

We may further study on this field by taking Mahmudov's work as a primary source. Next we can study the approximate controllability of quasilinear differential equations, as the quasilinear differential equations are frequently used in the traffic flow problems, continuum mechanics and it has many more applications in science and technology. This will generalize the existing results in this field.

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Weighted Shift Operators: a graph theoretical approach

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Abstract. In the branch of operator theory, weighted shift operators are of utmost importance. In this review, we discuss how some graph theoretical concepts are incorporated resulting in the introduction of a new class of weighted shift operators that are defined on a directed tree. We also state some important results on the adjacency operator of a directed as well as of an undirected graph.

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1 Introduction

The weighted shift operators are a vital class of linear operators that have been quite extensively studied till date. These interesting class of operators are of fundamental importance in the branch of operator theory. Some adequate references for weighted shifts are [5], [10], [11], [12], and [15].

Here, in this paper, we shall study weighted shift operators with a graph theoretic approach. An adequate reference in this regard is [6]. To begin with, we give a brief introduction of the weighted shift operator on a Hilbert space. Shift operators are basically of two types: the unilateral shift and the bilateral shift.

Let us consider a separable Hilbert space K and let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis of K. Then the unilateral shift is the operator that maps each orthonormal basis vector into a scalar multiple of the next basis vector. Let us consider a sequence of scalars $\{\alpha_n\}_{n=0}^{\infty}$. If $U : H \to H$ such that

$$Ue_n = \alpha_n e_{n+1} \text{ for } n = 0, 1, 2, \ldots$$

then U is called the unilateral shift operator on H and the α_n's are called the weights of the shift operator.

Its adjoint operator is given by

$$U^*e_0 = 0, \text{ and } U^*e_n = \bar{\alpha}_{n-1}e_{n-1} \text{ for } n = 1, 2, 3, \dots$$

Here, U^* is termed as the backward unilateral shift operator. When $\alpha_n = 1$ for all $n = 0, 1, 2, \dots$, the the weighted unilateral shift is called the unweighted unilateral shift.

To understand the shifts better, let us consider the sequence space $\ell^2_+(K)$.

$$\ell^2_+(K) = K \oplus K \oplus K \oplus \dots$$

is the space of all sequences $x = \{x_n\}_{n=0}^{\infty}$ of vectors $x_n \in K$. $\ell^2_+(K)$ is a Hilbert space with respect to the norm given by

$$\|x\|^2 = \sum_{n=0}^{\infty} \|x_n\|^2 < \infty.$$

The unilateral unweighted shift U_+ on $\ell^2_+(K)$ is defined as

$$U_+(x_0, x_1, \dots) = (0, x_0, x_1, \dots).$$

The multiplicity of U_+ is the cardinal number $n = \dim K$. It follows immediately that the adjoint of U_+ is given by

$$U^*_+(x_0, x_1, \dots) = (x_1, x_2, \dots)$$

and U^*_+ is called the backward shift. Two unilateral shift operators are unitarily equivalent if and only if they have the same multiplicity.

Again, let $\ell^2(K) = \sum_{-\infty}^{\infty} \oplus K$ be the Hilbert space of two-way sequences $x = (\dots, x_{-1}, [x_0], x_1, \dots)$ of vectors from K with $\|x\|^2 = \sum_{n=-\infty}^{\infty} \|x_n\|^2 < \infty$, and let the bilateral shift U on $\ell^2(K)$ be defined as

$$U(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, x_{-2}, [x_{-1}], x_0, \dots).$$

Here, $[\cdot]$ denotes the central 0th entry of $x = (\dots, x_{-1}, [x_0], x_1, \dots)$.
The multiplicity of U is $\dim K$. Just like unilateral shifts, two bilateral shifts are also unitarily equivalent if and only if they have the same multiplicity.

Motivated by the theory of unweighted unilateral and bilateral shifts, the class of operators namely weighted shift operators is defined on the Hilbert space $\ell^2(K)$. Here, we refer to the definition given by Shields in [15], where he considers weighted shifts on $\ell^2(K)$ for $\dim K = 1$. For a bounded sequence of scalars $\{\beta_n\}_{n\in\mathbb{Z}}$, the weighted unilateral shift S is defined on $\ell^2_+(K)$ as

$$S(x_0, x_1, \dots) = (0, \beta_0 x_0, \beta_1 x_1, \dots);$$

and the bilateral weighted shift W is defined on $\ell^2(K)$ as

$$W(\dots, x_{-1}, [x_0], x_1, \dots) = (\dots, \beta_{-2}x_{-2}, [\beta_{-1}x_{-1}], \beta_0 x_0, \dots).$$

In 1967, N. K. Nikolskii [10] introduced operator weighted shifts as a generalization of scalar weighted shifts. For a sequence of uniformly bounded operators $\{A_n\}_{n\in\mathbb{N}_0}$ on K, the operator T on $\ell^2_+(K)$, defined as

$$T(x_0, x_1, \dots) = (0, A_0x_0, A_1x_1, \dots)$$

is called the unilateral operator weighted shift with weights $\{A_n\}_{n\in\mathbb{N}_0}$. A bilateral operator weighted shift is similarly defined on $\ell^2(K)$.

2 Weighted shifts on graphs

Graph theoretical concepts have time and again been incorporated in the branch of operator theory. One such interesting and significant work is the study of weighted shift on directed trees. A significant paper in this context is [6]. In [6], the weighted shift taken into consideration is mainly motivated by two concepts: the weighted shift operator on a Hilbert space and the adjacency operator of a directed graph. Finite undirected graphs induce symmetric adjacency matrices while in case of infinite graphs, these matrices have to be replaced by adjacency operators. Adequate references for the study of adjacent operators are [9, 13, 14].

The notion of an adjacency operator for an infinite undirected graph was first introduced by Mohar in [9]. A locally finite countable graph $G = (V, E)$ is considered where $V = \{v_i : i \in \mathbb{N}\}$ is the set of vertices and E is the set of edges. $\mathbb{N}$ is the set of natural numbers. Then the associated (infinite) adjacency matrix $A(G) = [a_{ij}]$ is defined as the matrix where the (i, j)th entry a_{ij} is equal to the number of edges between the vertices v_i and v_j.

Mohar interpreted this adjacency matrix $A(G)$ with a linear operator on the Hilbert space ℓ^2 (or $\ell^2(\mathbb{C})$, where $\mathbb{C}$ is the set of complex numbers). Recall that ℓ^2 is the Hilbert space of all square summable complex sequences; i.e, it contains all the sequences of the form $(x_i : i \in \mathbb{N})$ such that $\sum_{i\in\mathbb{N}} |x_i|^2$ is convergent. Let $\{e_k = \delta_{ik} : i \in \mathbb{N}\}$ be an orthonormal basis of ℓ^2.

The adjacency matrix $A(G)$ of the locally finite undirected graph G can then be interpreted as the linear operator A on ℓ^2 such that

$$Ae_k = (a_{ik} : i \in \mathbb{N}), \text{ or equivalently}$$
$$\langle Ae_k, e_i\rangle = a_{ik}.$$

Since the graph G is locally finite, that means it has a finite number of edges adjacent to each vertex, so the operator A is well defined. Hence, Ae_k is in ℓ^2 and can be linearly extended to a dense subspace of ℓ^2 spanned by the basis vectors $\{e_k : k \in \mathbb{N}\}$. We denote this subspace as H_0 and A_0 be the corresponding linear operator on H_0. Since, the graph under consideration is undirected, so the operator A_0 is symmetric. Hence, A_0 is closable. The closure $A = \overline{A_0}$ of the linear operator A_0 is called the adjacency operator of the undirected graph G. Hence, we arrive at the following definition:

Definition 2.1. *[9] The adjacency operator is a closed symmetric transformation with domain $D(A)$, for which the following holds:*

$$\langle Ax, e_i\rangle = \sum_{j\in\mathbb{N}} a_{ij}x_j,\ i \in \mathbb{N},\ x \in D(A).$$

We shall next discuss the work done by Sasaoka in his paper [13], where the adjacency operator is defined on directed graphs. It should be noted that unlike in the case of undirected graphs, the adjacency operator here need not be self adjoint even if it is bounded. We begin with some basic definitions and notations on directed graphs.

Definition 2.2. *[13] A directed graph $G = (V, E, \delta^+, \delta^-)$ is a system of sets V, E and maps $\delta^{\pm} : E \longrightarrow V$. Here,*
- *Elements of V are the vertices.*
- *Elements of E are the edges.*
- *For an edge $e \in E$, $\delta^+(e)$ is an initial vertex and $\delta^-(e)$ is a terminal vertex.*
- *For each vertex $v \in V$, the outdegree, $d^+(v) =$ cardinality of the set $\{e \in E : \delta^+(e) = v\}$*
- *the indegree, $d^-(v) =$ cardinality of the set $\{e \in E : \delta^-(e) = v\}$.*
- *The valency (or degree), $d(v) = d^+(v) + d^-(v)$.*

Definition 2.3. *[13] If the valency $d(v)$ is finite for every $v \in V$, then the graph G is a locally finite graph. A graph has bounded valency if there is a constant $M > 0$ such that $d(v) \leq M$ for any vertex $v \in V$.*

For an edge $e \in E$, let $\delta^+(e) = u$ and $\delta^-(e) = v$. Then u is called a server of v and v is called a receiver of u. If a vertex w is a server of two vertices u and v, then w is called the common server of u and v. Similarly w is called a common receiver of u and v, if w is a receiver of u and v . We denote the number of all common servers of u and v by $d^+(u, v)$ and common receivers as $d^-(u, v)$.

Definition 2.4. *[13] We define the following:*
$D^+(v) = u \in V : u$ *is a receiver of* v,
$D^-(v) = u \in V : u$ *is a server of* v,
$D^+(u, v) = w \in V : w$ *is a common receiver of* u, v, *and*
$D^-(u, v) = w \in V : w$ *is a common server of* u, v.

Sasaoka considered the infinite directed graph G to be locally finite and without multiple edges between any two vertices. We proceed to define the adjacency operator on such a graph G. We consider a Hilbert space $H = l^2(V)$ with the canonical basis $\{e_v : v \in V\}$ defined as $e_v(u) = \delta_{v,u}$ for $u, v \in V$. Let H_0 be the linear span of the set $\{e_v : v \in V\}$. Let A_0 and B_0 be two linear operators on H with the dense domains $Dom(A_0) = H_0 = Dom(B_0)$. For an element $\sum_{v\in V} x_v e_v \in H_0$, the operators A_0 and B_0 are defined as

$$A_0(\sum_{v\in V} x_v e_v) = \sum_{u\in V} \sum_{v\in D^-(u)} x_v e_u, \text{ and}$$
$$B_0(\sum_{v\in V} x_v e_v) = \sum_{u\in V} \sum_{v\in D^+(u)} x_v e_u.$$

The operators A_0 and B_0 are well-defined since the graph G is locally finite. A_0 and B_0 are closable and $A_0^* \supset \overline{B_0}$, and $B_0^* \supset \overline{A_0}$.

Definition 2.5. *[13] Let $A = A(G)$ be a closed operator with the domain $Dom(A)$ given by*

$$Dom(A) = \{x = \sum_{v\in V} x_v e_v \in H : \sum_{u\in V} | \sum_{v\in D^-(u)} x_v|^2 < \infty\}$$

Then, for $x \in Dom(A)$, the operator $A = A(G)$ defined as

$$Ax = \sum_{u \in V} \sum_{v \in D^-(u)} x_v e_u$$

is called the adjacency operator of the directed graph G .

Similarly, a closed operator B with the domain $Dom(B)$ is defined as

$$Bx = \sum_{u \in V} \sum_{v \in D^+(u)} x_v e_u,$$

where

$$Dom(B) = \{x = \sum_{v \in V} x_v e_v \in H : \sum_{u \in V} | \sum_{v \in D^+(u)} x_v|^2 < \infty\}.$$

Sasaoka proved in his paper that the adjacency operator is bounded if and only if the graph G has bounded valency. He also proved several significant results on the adjacency operator in relation with the graph G.

Motivated by [13], Jablon'ski, Jung and Stochel introduced weighted shifts on directed trees in [6]. The weighted shift considered here can be interpreted as the generalization of classical weighted shifts and adjacency matrices. The results in [6] aim to show the similarity between classical weighted shifts and the shifts defined on trees, and also point out few advantages of the later over the former in some cases. We start with some preliminary definitions and notations given in [6]:

Let $G = (V, E)$ be a directed graph, where V is a non empty set of vertices and E is the set of directed edges, i.e, E is a subset of $V \times V \setminus \{(v, v) : v \in V\}$. Let the set $\tilde{E}$ represents the set of undirected edges i.e,

$$\tilde{E} = \{\{u, v\} \subseteq V : (u, v) \in E \text{ or } (v, u) \in E\}.$$

For a non empty subset W of V, the graph $G_W = (W, (W \times W) \cap E)$ is called a directed subgraph of G.

Definition 2.6. *[6] A directed graph G is said to be connected if for any two distinct vertices u and v of G, there exists a finite sequence $v_1, v_2, \ldots, v_n$ of vertices of G, where $n > 2$ such that $u = v_1$, $\{v_j, v_{j+1}\} \in \tilde{E}$ for all $j = 1, 2, \ldots, n-1$, and $v_n = v$. Such a sequence is called an undirected path joining u and v.*

Definition 2.7. *[6] • For a vertex u in V, a member of the set $Chi(u)$ is called a child of u, where*

$$Chi(u) = \{v \in V : (u, v) \in E\},\ u \in V.$$

• If for a given vertex $u \in V$, there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then u is said to have a parent v and then v is written as $par(u)$.

• *A finite sequence* $\{u_j\}_{j=1}^{n}$ *of distinct vertices is said to be a circuit of* G *if* $n \geq 2$, $(u_j, u_{j+1}) \in E$ *for all* $j = 1, 2, \ldots, n-1$, *and* $(u_n, u_1) \in E$.

• *A vertex* v *of* G *is called a root of* G, *i.e,* $v \in Root(G)$, *if there is no vertex* $u \in G$ *such that* $(u, v) \in E$. *Clearly, we can have more than one of such roots. If there is only one root, then we write the unique root as* $root(G)$.

In fact the directed graph G has a unique root if it is connected and each of its vertices other than the root has a parent. With this result, we arrive at the definition of a directed tree:

Definition 2.8. *[6] A directed graph* $\mathcal{T}$ *is said to be a directed tree if it has no circuits and satisfies the following two conditions:*
(i) $\mathcal{T}$ *is a connected graph*
(ii) Each vertex other than the root has a parent.

A subgraph of $\mathcal{T}$ is a subtree of $\mathcal{T}$ if it itself is a directed tree. Note that, a directed tree $\mathcal{T}$ may not necessarily have a root, but if it does, the root has to be unique. Since the set V of vertices may also be finite, so there is a possibility of a finite directed tree. In such cases, the root is always unique. We denote V^0 as the set of vertices without the roots,i.e, $V^0 = V \setminus Root(G)$. The following proposition shows a decomposition of the set V^0:

Proposition 2.9. *[6] If* $\mathcal{T}$ *is a directed tree, then for any* $u, v \in V$, $u \neq v$ *we must have* $Chi(u) \cap Chi(v) = \phi$, *and* $V^0 = \bigsqcup_{u \in V} Chi(u)$.

For any subset W of V, we can write $Chi(W) = \sum_{v \in W} Chi(v)$ in view of the above proposition. We define the following:

$$(i) Chi^{(0)}(W) = W,$$
$$(ii) Chi^{(n+1)}(W) = Chi(Chi^{(n)}(W)),\ n = 0, 1, \ldots,$$
$$(iii) \text{Descendents of } W, Des(W) = \bigcup_{n=0}^{\infty} Chi^{(n)}(W).$$

Since by definition the directed tree $\mathcal{T}$ has no circuits, so the sets $Chi^{(n)}(u)$, $n - 0, 1, 2 \ldots$ are pairwise disjoint. Hence we can decompose the set $Des(u)$ as

$$Des(u) = \bigsqcup_{n=0}^{\infty} Chi^{(n)}(u),\ u \in V.$$

It is very interesting to see that if the directed tree $\mathcal{T}$ has a *root*, then the whole set of vertices V can be obtained as

$$V = Des(root).$$

The following result shows us a decomposition of the directed tree $\mathcal{T}$:

Proposition 2.10. *[6] Let* $\mathcal{T}$ *be a directed tree and* $u \in V$. *Then* $\mathcal{T}$ *can be decomposed into two different subtrees* $\mathcal{T}_{Des(u)}$ *and* $\mathcal{T}_{V \setminus Des(u)}$, *which are given as*
(i) $\mathcal{T}_{Des(u)}$ *is a directed tree with root* u.

(ii) $\mathcal{T}_{V\setminus Des(u)}$ is a directed tree, provided $V \setminus Des(u) \neq \phi$. If $\mathcal{T}$ has a root, then so does $\mathcal{T}_{V\setminus Des(u)}$ and in fact $root(\mathcal{T}) = root(\mathcal{T}_{V\setminus Des(u)})$.
Moreover, if $Des(u) = Des(v)$, then we must have $u = v$.

We now move on to discuss the weighted shift that is defined on such a directed tree. Let us consider $\ell^2(V)$ as the space of all square summable complex functions on the set of vertices V of the directed tree $\mathcal{T} = (V, E)$. $\ell^2(V)$ is a Hilbert space with the standard inner product

$$\langle f, g\rangle = \sum_{u\in V} f(u)\overline{g(u)}.$$

The set $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$, where the element $e_u \in \ell^2(V)$ is given by

$$e_u(v) := \begin{cases} 1, & \text{if } u = v; \\ 0, & \text{otherwise.} \end{cases}$$

Let ε_V be the linear span of the set $\{e_u\}_{u\in V}$. In fact, $\ell^2(V)$ is a reproducing kernel Hilbert space with the reproducing property given by

$$f(u) = \langle f, e_u\rangle$$

for $f \in \ell^2(V)$ and $u \in V$.

Definition 2.11. *[6] Given $\lambda = \{\lambda_v\}_{v\in V^0}$, a family of complex numbers, we define the operator S_λ in $\ell^2(V)$ by*

$$\mathcal{D}(S_\lambda) = \{f \in \ell^2(V) : \Lambda_{\mathcal{T}} f \in \ell^2(V)\}$$
$$S_\lambda f = \Lambda_{\mathcal{T}} f,\ f \in \mathcal{D}(S_\lambda),$$

where $\Lambda_{\mathcal{T}}$ is the mapping defined on functions $f : V \longrightarrow \mathbb{C}$ by

$$(\Lambda_{\mathcal{T}} f)v := \begin{cases} \lambda_v . f(par(v)), & \text{if } v \in V^0; \\ 0, & \text{if } v = root. \end{cases}$$

The operator S_λ is called a weighted shift on the directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^0}$.

Proposition 2.12. *[6] Let S_λ be a weighted shift on a directed tree $\mathcal{T}$. Then following are some of the important assertions that hold for S_λ:*
(i) S_λ is a closed operator.
(ii) $\mathcal{D}(S_\lambda) = \{f \in \ell^2(V) : \sum_{u\in V}(\sum_{v\in Chi(u)} |\lambda_v|^2)|f(u)|^2 < \infty\}$.
(iii) $\|f\|^2_{S_\lambda} = \sum_{u\in V}(1 + \sum_{v\in Chi(u)} |\lambda_v|^2)|f(u)|^2$ for all $f \in \mathcal{D}(S_\lambda)$.
(iv) $e_u \in \mathcal{D}(S_\lambda)$ if and only if $\sum_{v\in Chi(u)} |\lambda_v|^2$ is finite. If $e_u \in \mathcal{D}(S_\lambda)$, then

$$S_\lambda e_u = \sum_{v\in Chi(u)} \lambda_v e_v, \text{ and}$$
$$\|S_\lambda e_u\|^2 = \sum_{v\in Chi(u)} |\lambda_v|^2.$$

(v) S_λ is densely defined if and only if $\{e_u : u \in V\} \subseteq \mathcal{D}(S_\lambda)$.
(vi) $S_\lambda = \overline{S_\lambda|_{\varepsilon_V}}$ if S_λ is densely defined.

For a densely defined weighted shift S_λ, the linear space ε_V is an invariant subspace only when the set $Chi(u)$ is infinite for at least one vertex u, and all the weights $\{\lambda_v\}_{v\in Chi(u)}$ are non zero. However, another condition for ε_V to be invariant is that the set $Chi(u)$ must be finite for every vertex u in V.

Now we shall see how exactly we can compare a classical weighted shift on a Hilbert space with a weighted shift on a directed tree. Let us consider the directed tree $(\mathbb{Z}_+, \{n, n+1\} : n \in \mathbb{Z}_+)$, where $\mathbb{Z}_+$ is the set of non negative integers. Then for $n \in \mathbb{Z}_+$, $Chi(n) = n+1$. So, from (iv) of the above proposition, we get

$$S_\lambda e_n = \sum_{v\in Chi(n)} \lambda_v e_v$$
$$\Rightarrow S_\lambda e_n = \lambda_{n+1} e_{n+1},$$

which is nothing but the unilateral weighted shift operator. This may slightly differ from the usual notation $S_\lambda e_n = \lambda_n e_{n+1}$, but this does not cause any problem to the work. Also, for each $n \in \mathbb{Z}_+$, $e_n \in \mathcal{D}(S_\lambda)$ since $\sum_{v\in Chi(n)} |\lambda_v|^2 = |\lambda_{n+1}|^2$. Hence, any weighted shift S_λ on the directed tree $(\mathbb{Z}_+, \{n, n+1\} : n \in \mathbb{Z}_+)$ is densely defined. Also here the linear span of $\{e_n : n \in \mathbb{Z}_+\}$ is a core of S_λ and so it is justified that S_λ is a unilateral classical weighted shift.

Similarly by considering the directed tree $(\mathbb{Z}, \{n, n+1\} : n \in \mathbb{Z})$, where $\mathbb{Z}$ is the set of integers, we can arrive at the definition of a classical bilateral weighted shift.

The weighted shift S_λ can be decomposed as an orthogonal sum of two weighted shifts on two different trees. This can be stated as the following proposition:

Proposition 2.13. *[6] Let S_λ be a weighted shift on a directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^0}$. Assume that $\lambda_u = 0$ for some $u \in V^0$. Then*

$$S_\lambda = S_{\lambda\to(u)} \oplus S_{\lambda\leftarrow(u)}.$$

Here, $S_{\lambda\to(u)}$ is a weighted shift on the directed tree $\mathcal{T}_{Des(u)}$ with weights $\lambda_\to(u) := \{\lambda_v\}_{v\in Des(u)\setminus\{u\}}$, and $S_{\lambda\leftarrow(u)}$ is a weighted shift on the directed tree $\mathcal{T}_{V\setminus Des(u)}$ with weights $\lambda_\leftarrow(u) := \{\lambda_v\}_{v\in V\setminus(Des(u)\cup root(\mathcal{T}))}$.

The injectivity of the weighted shift S_λ depends mainly of the directed tree $\mathcal{T}$ on which it is defined. Interestingly, it may happen that S_λ is an injective map even when some of the weights $\{\lambda_v\}_{v\in V^0}$ are taken to be zero. But for a classical weighted shift to be injective, each of the weights must always be non zero. The following result gives us a necessary and sufficient condition for S_λ to be injective.

Proposition 2.14. *[6] Let S_λ be a weighted shift on a directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^0}$. Then the following conditions are equivalent:*
(i) S_λ is injective,
(ii) $\mathcal{T}$ is leafless and $\sum_{v\in Chi(u)} |\lambda_v|^2 > 0$ for every $u \in V$. By a leafless tree, we mean that each $u \in V$ must have at least one child, i.e, the set $Chi(u)$ must be non empty.

The following proposition gives us a simple condition for the weighted shift S_λ to be bounded.

Proposition 2.15. *[6] Let S_λ be a weighted shift on a directed tree $\mathcal{T}$ with weights $\{\lambda_v\}_{v\in V^0}$. Then the following conditions are equivalent:*
(i) $\mathcal{D}(S_\lambda) = \ell^2(V)$,
(ii) S_λ is a bounded linear operator on $\ell^2(V)$,
(iii) $\sup_{u\in V}\sum_{v\in Chi(u)}|\lambda_v|^2 < \infty$.
If S_λ is bounded, then its norm is given by

$$\|S_\lambda\| = \sup_{u\in V}\|S_\lambda e_u\| = \sup_{u\in V}\sqrt{\sum_{v\in Chi(u)}|\lambda_v|^2}.$$

Another advantage of the weighted shift on trees over classical weighted shifts is that it is possible to construct a reducible injective and bounded weighted shift on a directed tree with root. But classical weighted shifts are never reducible. Also, we know that if U is a classical weighted shift with weights $\{w_n\}_{n\in\mathbb{N}_0}$, then U is unitarily equivalent to a weighted shift operator with weight sequence $\{|w_n|\}_{n\in\mathbb{N}_0}$. We have an analogous result in the case of S_λ.

Theorem 2.16. *[6] A weighted shift S_λ on a directed tree $\mathcal{T}$ with weights $\lambda = \{\lambda_v\}_{v\in V^0}$ is unitarily equivalent to the weighted shift $S_{|\lambda|}$ on $\mathcal{T}$ with weights $|\lambda| = \{|\lambda_v|\}_{v\in V^0}$.*

Next we shall discuss about the adjoint operator of the weighted shift S_λ, and also show that S_λ can be interpreted as the adjoint of a classical weighted shift.

Proposition 2.17. *[6] Let S_λ be a densely defined weighted shift on a directed tree $\mathcal{T}$ with weights $\lambda = \{\lambda_v\}_{v\in V^0}$. Then the following assertions hold:*
(i) $\sum_{v\in Chi(u)}|\lambda_v f(v)| < \infty$ for all $u \in V$ and $f \in \ell^2(V)$,
(ii) $\varepsilon_V \in \mathcal{D}(S_\lambda^*)$ and

$$S_\lambda^* e_u := \begin{cases} \overline{\lambda_u}.e_{par(u)}, & \text{if } u \in V^0; \\ 0, & \text{if } u = root. \end{cases}$$

(iii) $(S_\lambda^* f)(u) = \sum_{v\in Chi(u)}\overline{\lambda_v} f(v)$ for all $u \in V$ and $f \in \mathcal{D}(S_\lambda^*)$,
(iv) $\mathcal{D}(S_\lambda^*) = \{f \in \ell^2(V) : \sum_{u\in V}|\sum_{v\in Chi(u)}\overline{\lambda_v} f(v)|^2 < \infty\}$.
(v) $\|f\|^2_{S_\lambda^*} = \sum_{u\in V}(|f(u)|^2 + |\sum_{v\in Chi(u)}\overline{\lambda_v} f(v)|^2)$ for all $f \in \mathcal{D}(S_\lambda^*)$,
(vi) $\ell^2(Chi(u)) \subseteq \mathcal{D}(S_\lambda^*)$ for all $u \in V$,
(v) $S_\lambda^* = \overline{S_\lambda^*|_{\varepsilon_V}}$.

Unlike in the case of S_λ, ε_V is always an invariant subspace for the adjoint S_λ^* of a densely defined weighted shift S_λ. Finally, we shall show that the adjoint of a unilateral classical weighted shift is in fact a weighted shift that is defined on a specific directed tree. Let us consider the subtree $(\mathbb{Z}_-, \{n, n+1\} : n \in \mathbb{Z}_-)$ of the directed tree $(\mathbb{Z}, \{n, n+1\} : n \in \mathbb{Z})$. Clearly the subtree $(\mathbb{Z}_-, \{n, n+1\} : n \in \mathbb{Z}_-)$ has no root and has only one leaf 0 (A vertex $u \in V$ is called a leaf if it has no child,i.e, $Chi(u) = \phi$). Let S_λ be a weighted shift on $(\mathbb{Z}_-, \{n, n+1\} : n \in \mathbb{Z}_-)$ with weights $\lambda = \{\lambda_{-n}\}_{n=0}^\infty$. In view of Proposition 2.12, S_λ is densely defined, and is given by

$$S_\lambda e_{-n} = \lambda_{-(n-1)} e_{-(n-1)} \text{ for all } n = 1, 2, \ldots, \text{ and } S_\lambda e_0 = 0.$$

The assertion (vi) of the Proposition 2.12 ensures that S_λ can be interpreted as the adjoint of the classical unilateral weighted shift with weights $\{\bar{\lambda}_{-(n-1)}\}_{n=1}^\infty$. Again, we know that the classical

unilateral weighted shift is an isometry if the weight sequence is considered as the constant sequence of 1. In the present case, S_λ defined on directed tree $\mathcal{T}$ is an isometry on $\ell^2(V)$ if and only if $\sum_{v \in Chi(u)} |\lambda_v|^2 = 1$ for all $u \in V$.

Hence, we have seen in [6] as how the structure of a graph helps in the study of the weighted shift S_λ. In the same paper, several other significant properties of the weighted shift S_λ such as hyponormality, cohyponormality and subnormality have been discussed in sufficient detail. For further knowledge in this context, one may refer to the following: [1, 2, 3, 4].

3 Conclusion

It is very interesting to see the shift operator evolve with a completely different approach while incorporating concepts of graph theory in the process. One can always compare the various properties of the classical weighted shift on a Hilbert space with those of the weighted shift defined on a directed tree. Till date, various significant work has been done in this context. However, there is always a scope of further research in this area.

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Estimation of Value at Risk, Expected Shortfall and Median Shortfall

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Abstract. Value-at-risk (VaR), Expected shortfall (ES) and Median shortfall (MS) are the well known risk measures to estimate the market risk. VaR is essentially an extreme quantile and is a popular risk measure to estimate market risk. However there are several demerits of VaR. ES is the mean of the conditional return distribution, given the event that the return is less than the VaR and MS is the median of the conditional return distribution, given the event that the return is less than the VaR. Estimation of these risk measures is an important problem in finance. In this paper we discuss various estimators of these risk measures and compare their performance using Monte Carlo simulation.

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1 Introduction

Risk measures have become important tools in finance and actuarial science. A risk measure is a function that assigns real numbers to the possible outcomes of a random financial quantity, such as an insurance claim or loss of a portfolio (see [10]). Market risk of a portfolio refers to the loss incurred due to random fluctuations in the value of risky assets in the portfolio over a period of time. Value-at-risk (VaR) is a popular measure of market risk associated with an asset or a portfolio of assets. Its use was recommended by the Basel Committee in 1996 (Basel Committee on Banking Supervision (1996)) and also in the latest proposed Basel II standards (Basel Committee on Banking Supervision (2003)). It is defined as follows.

Let, X be a random variable representing a loss of some financial position. For $0 < p < 1$, the $(1-p)$th quantile of the distribution with distribution function F is defined as

$$Q_p = \inf\{x : F(x) \geq (1-p)\},$$

the $100(1-p)$ percent VaR, denoted by VaR_p, is the negative $(1-p)$th quantile of the marginal distribution of X, i.e.

$$VaR_p = -Q_p. \tag{1.1}$$

Hence estimation of VaR_p essentially reduce to the problem of estimation of the quantile Q_p.

Artzner et al. introduced the concept of coherent risk measures (see [4, 5]). They argued that a risk measure should satisfy four desirable properties: monotonicity, subadditivity, positive homogeneity and translation invariance. Artzner et al. [7] pointed out that VaR is not a coherent risk measure because it does not satisfy the subadditivity condition. This implies that the risk of a portfolio, when measured by VaR, can be larger than the sum of the standalone risks of its components.

To construct a risk measure that is both coherent and easy to compute and estimate, the expected shortfall (ES) was proposed and discussed by Artzner et al. [5]. For a given level p, the shortfall distribution is given by the cumulative distribution function Θ_p defined by:

$$\Theta_p(x) = P\{X \leq x | X > VaR_p\}$$

This distribution is just the conditional loss distribution i.e the truncated distribution, given that the loss exceeds the VaR at the same level. The mean of this distribution is called the expected shortfall, and is denoted by ES_p. Mathematically, it can be written as

$$ES_p = -\frac{1}{p}\int_{VaR_p}^{\infty} x dF(x)$$

Among all coherent measures, expected shortfall is regarded as a good supplement to VaR, as it is closely linked to VaR (see [1, 2, 3]).

So and Wong introduced the risk measure called Median Shortfall (MS) (see [31]). By definition, MS is the median of the conditional return distribution, given that the return is less than the VaR level (see [31]). Let Θ_p denote the distribution function of this conditional return distribution. It is defined as follows

$$\Theta_p(x) = P\{X \leq x | X > Q_p\}.$$

The median of this distribution is called the Median Shortfall, denoted by MS_p (see [31]). The MS can be written as

$$MS_p = -\inf\{x : \Theta_p(x) \geq 0.5\} = -Q_{0.5p}. \tag{1.2}$$

From the definition it is clear that median shortfall is nothing but the value-at-risk at some higher level. As we know that there are some disadvantages of mean with respect to median, the performance of the Median Shortfall is compared with that of Expected Shortfall.

In this paper we discuss several estimators of VaR, ES and MS and compare their performance using Monte Carlo simulations. In section 2 we give a brief literature review. In section 3 we review the estimators of VaR and MS. In section 4 we review the estimators of ES. In section 5 we discuss about the simulation study and in section 6 is the conclusion.

2 Literature Review

As VaR is essentially an extreme quantile, thus the VaR estimation reduces to the problem of estimation of extreme quantile. The methods of quantile estimation fall into two broad categories: parametric and non-parametric techniques. In parametric technique we specify a distribution of the returns. But in non-parametric technique no such assumption is imposed on the underlying distribution of the returns.

Early estimators of VaR are based on parametric models for the return distribution F, for instance, Gaussian or t-distribution. More sophisticated parametric approach based on autoregressive conditional heteroskedastic (ARCH) or generalized ARCH (GARCH) models has been developed under the trademarks of RiskMetrics, KMV and Creditmetrics. The advantages of the parametric approaches lay in their easy interpretation. But they are model dependent and are subject to errors of model misspecification.

Again in actuarial science and financial risk management extreme quantile focuses only on downside risks, practitioners understood that their aim should be on fitting tails of distributions, i.e. skewed and heavy tailed distributions. In Gencay and Selcuk [18], Matthys and Beirlant [22], McNeil [23], McNeil et al. [24], Embrechts et al. [17] and Charpentier and Oulidi [12], authors use the Extreme Value Theory (EVT) or exactly the peaks over threshold (POT) method for modeling tails of loss distributions and for estimating extreme quantiles. But this approach leads to particular parametric models, as from Pickands-Balkema-de Haan theorem [9] tails should be either Pareto type or exponential type. Again a misspecification of the model can induce substantial errors.

So in order to have a "distribution free" nature, non-parametric models are used to estimate extreme quantiles. A traditional estimator of the $(1-p)$th quantile of a random variable X is the pth sample quantile. The main drawback to sample quantiles is that they experience a substantial lack of efficiency, caused by the variability of individual order statistics. The efficiency of the sample quantile can be improved by considering linear combinations of order statistics, that is, L-estimators (see [30]). Another approach is based on numerical inversion techniques of nonparametric kernel estimators of the cdf as the quantile function is simply the inverse of the cumulative distribution function which was discussed by Azzalini [8]. In the kernel method the main problem lies with the selection of bandwidth. Now with a proper choice of bandwidth one can expect upto 15% improve in efficiency over the usual sample quantile. Chen and Tang [13], Azzalini [8] and Bowman [11] provide some choice of bandwidth parameter.

But Artzner et al. [6] showed that VaR has several shortcomings and introduced a new measure of risk referred to as the ES. Several estimators of ES, both parametric and non-parametric are reviewed by Nadarajah et al. [26]. They mentioned that the empirical estimator is the best method for estimating ES. From the definitions we can see that both VaR and ES are closely related to each other. In spite of overcoming the shortcomings of VaR, ES has its own shortcomings.

We know that expected shortfall is the mean of the truncated distribution and mean has a lot of disadvantages. As we know that mean is very susceptible to outliers, while the median is not affected by outliers. So we use MS which is the median of the truncated distribution. In the next section we review the quantile estimators.

3 Quantile estimators

Quantile estimation based on Extreme Value Theory

Estimation of quantiles for values of p close to 1 by extreme value theory is related to Pickands-Balkema-de Haan Theorem (see [9]). If $X_1, X_2,\ldots,X_n$ are independent and identically distributed with distribution function F and $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denotes the associated order statistics then the limiting distribution of an affine transformation is $X_{(n)}$ is either Fréchet, Weibull or Gumbel. The shape of the underlying distribution F fully characterizes the limiting distribution (called max-domain of attraction). The limiting distribution is characterized by a tail index ξ, using the Generalized Extreme Value representation (GEV). If $\xi > 0$ the limiting distribution is a Fréchet distribution, if $\xi = 0$ the limiting distribution is a Gumbel distribution and if $\xi < 0$ the limiting distribution is a Weibull distribution. Pickands-Balkema-de Haan Theorem claims that if F is in the max-domain of attraction of the Generalized Extreme Value (GEV) distribution of parameter ξ, for u large enough, $X - u$ given that $X > u$, has a Generalized Pareto distribution with tail parameter ξ and with some shape parameter $\beta(u)$.

Based on this result we discuss an estimator called GPD estimator. If $\xi > 0$, and if $\hat{\xi}$ and $\hat{\beta}$ denote the maximum likelihood estimates of the Pareto distribution, based on the pseudo sample $\{X_{(n-k+1)} - X_{(n-k)}, \ldots, X_{(n)} - X_{(n-k)}\}$. then

$$EV_p = X_{(n-k)} + \frac{\hat{\beta}_k}{\hat{\xi}_k}\left(\left[\frac{n}{k}(1-p)\right]^{(-\hat{\xi}_k)} - 1\right)$$

When $\frac{n}{k} \approx 1$, this estimator is called the GPD estimator (see [12]). The main idea is to estimate a parametric model on subsample, taking into account only the largest values so that the asymptotic properties remain in the tail when considering the subsample. Another main constraint is to have a sample large enough, because we need many observations in order to use the maximum-likelihood technique. An important issue is the choice of threshold u, i.e. of k (see [20]).

Non-parametric technique

In non-parametric quantile estimation, we have traditional estimator called the sample quantile. Let $I(\cdot)$ be the indicator function, with $I(S)$ equal to 0 or 1 according as the statement S is false or true. If $\hat{F}(x) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq x)$ i.e. $\hat{F}$ is the empirical distribution function, $\hat{Q}_p$ equals $-X_{([n(1-p)]+1)}$, where $\lfloor x \rfloor$ denotes the integral part of x. It is the pth sample quantile (we call it SQ_p).

Kernel method for estimating quantile

Various authors like Nadaraya [25], Azzalini [8], Gouriéroux et al. [19] and Chen and Tang [13] have discussed about the kernel method for estimating quantile.

Definition 3.1. *The kernel based estimate of F, for sample $\{X_1, X_2,\ldots,X_n\}$ is*

$$\tilde{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n}\int_{-\infty}^{t} w\left(\frac{x - X_i}{b}\right)$$

where w is the kernel function and b is the bandwidth. With a proper choice of bandwidth one can expect upto 15% improve in efficiency over the usual sample quantile. Chen and Tang [13], Azzalini [8] and Bowman [11] provide some choice of bandwidth parameter. Chen and Tang [13] have obtained the asymptotic bias, variance and the rate of almost sure convergence of their version of $\hat{Q}_p$, under the assumption that $\{X_t\}$ is a stationary geometric $\alpha-$mixing process. The authors suggested the following choice for the optimal value of b,

$$b_{opt1} = \left\{ \frac{2f^3(Q_p)b_k}{\sigma_k^4(f^{(1)}(Q_p))^2} \right\}^{1/3} n^{-1/3},$$

where $b_k = \int uw(u)G(u)du$, and $\sigma_k^2 = \int u^2 w(u)du$. $G(\cdot)$ is the distribution function of the distribution with density w. b_{opt1} involves unknown constants Q_p, f and its derivative $f^{(1)}$ at Q_p. Chen and Tang [13] suggested to approximate Q_p in b_{opt1} by the corresponding sample quantile. The authors suggested to approximate f and $f^{(1)}$ by the density and the first derivative of the generalized Pareto distribution. We denote the Chen and Tang s' quantile estimator by C-T$_p$. We even propose a new method for data based selection of bandwidth. We use the Bootstrap method to estimate our proposed bandwidth. Azzalini [8] proposed the following bandwidth $b_{opt2} = Kn^{-c}$, where $K = 1.5\sigma$, n is the sample size and $c = 1/3$. We denote the Azzalini's quantile estimator by A_p. Bowman [11] proposed the following bandwidth $b_{opt3} = Kn^{-c}$, where $K = 3.9\sigma$, n is the sample size and $c = 1/3$. We denote the Bowman's quantile estimator by B_p.

L-statistics for estimating quantile

"An obvious way of improving the efficiency of sample quantiles is to reduce this variability by forming a weighted average of all the order statistics, using an appropriate weight function" [30]. These estimators are commonly called L estimators. A popular class of L estimators is called the kernel quantile estimators.

Definition 3.2. *The kernel quantile estimator is given by*

$$SM_p = -\sum_{i=1}^{n} \left[\int_{\frac{i-1}{n}}^{\frac{i}{n}} w\left(\frac{t-p}{b}\right) dt \right] X_{(i)}$$

where w is the kernel and b is the bandwidth. Several estimators have been considered, based on this expression, with different choice of w and b. Parzen [28], Padgett [27] and Sheather and Marron [30] have considered Gaussian kernels.

4 Non-parametric methods for estimating expected shortfall

In this section we review the non parametric estimators of ES.

Empirical estimator

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics in ascending order corresponding to the original data $X_1, X_2, \ldots, X_n$. The empirical estimator suggests to estimate expected shortfall by

$$Emp_p = -\frac{\sum_{i=[np]}^{n} X_{(i)}}{n - [np]}$$

where $[x]$ denotes the largest integer not greater than X. This method is considered as the best method of estimation for expected shortfall (see [26]).

Brazauskas et al.'s estimator

Let us recall that expected shortfall is defined as

$$ES_p = -\frac{1}{p}\int_{1-p}^{1} Q(u)du.$$

Let $\hat{F}$ denote the empirical cumulative distribution function of $X_1, \cdots, X_n$ and $\hat{F}^{-1}$ be its quantile function. Brazauskas et al. [10] defined an empirical estimator of ES_p as follows

$$\widehat{ES}_p = -\frac{1}{p}\int_{1-p}^{1} \hat{F}^{-1}(u)du.$$

Under the assumption that $X_1, \cdots, X_n$ are i.i.d. with $E|X_1| < \infty$, $\widehat{ES}_p$ converges to ES_p almost surely as n is increased (see [**?**]).

Yamai and Yoshiba's estimator

Yamai and Yoshiba [32] defined the following estimator of ES_p

$$ES_{p,\beta} = -\frac{1}{n(\beta+1+p)}\sum_{i=[n(1-p)]}^{n\beta} X_{(i)},$$

where β is a positive constant such that $X_{(1)} < X_{(2)} < .... < X_{[n(1-p)]} < .... < X_{(n\beta)} < .... < X_{(n)}$.

Kernel method

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ denote the order statistics in ascending corresponding to the financial returns X_1, X_2,...,X_n. Let K(.) denote a symmetric kernel, b a suitable bandwidth, $K_b(u) = \frac{1}{h}K(\frac{u}{h})$, $A(x) = \int_{-\infty}^{x} K(u)du$ and $A_b(u) = A(\frac{u}{b})$. Yu et al. [33] suggest various formulas for kernel estimation of expected shortfall and one of them is

$$ESker_p = -\frac{1}{np}\sum_{i=1}^{n} X_i A_b(\hat{q}(p) - X_i),$$

where

$$\hat{q}(p) = \sum_{i=1}^{n}\left[\int_{i-\frac{1}{n}}^{\frac{1}{n}} K_b(t-p)dt\right] X_{(i)}.$$

An alternative is to obtain $\hat{q}_p$ as the solution of

$$\frac{1}{n}\sum_{i=1}^{n} A_b(x - x_i) = p.$$

Comprehensive discussions of this kernel method are available in Scaillet [29] and Chen [14].

Jadhav et al.'s estimator

Jadhav et al. [21] propose several modifications of the empirical estimator for expected shortfall. One proposed estimator is

$$Ja_p = \frac{\sum_{i=0}^{[np^{1+a}]+1} X_i}{[np^{1+a}]+2},$$

where $i = [(n+1)p'_{(i)}]$, $p'_{(i)} = p - \frac{ip}{[np]+1}$, $i = 0, 1, \ldots, [np^{1+a}]+1$ and a is a constant taking values in [0, 0.1].

5 Simulation study

Comparing quantile estimators

In this section we study the performance of seven quantile estimators viz: EV_p, SQ_p, $C-T_p$, A_p, B_p and SM_p. We use the mean square error (MSE) of each of the mentioned estimators as a measure of performance. The mean square error is defined as

$$\begin{aligned} MSE &= E[(\hat{Q}(X,p) - Q)^2] \\ &\approx \frac{1}{m}\sum_{k=1}^{m}(\hat{Q}_n^{(k)}(p) - Q(p))^2 \end{aligned}$$

where $\hat{Q}_n^{(k)}(p)$ is the quantile obtained on the kth simulated sample and n is the sample size. The sample sizes considered are $n = 30$, 100, 250, 500, 1000 and 2500. Computations have been obtained by computing $m = 500$ samples from each distribution. We ran our simulations for distributions like standard Normal, standard Cauchy, Weibull, Lognormal and GPD. We even ran our simulations for dependent cases. The models considered are

1. AR(1) model: $X_t = aX_{t-1} + \sqrt{1-a^2}Z_t$, $Z_t \overset{iid}{\sim} \mathrm{N}(0, 1)$ and $a = 0.08$.
2. Log model: $log(X_t) = alog(X_{t-1}) + \sqrt{1-a^2}Z_t$, $Z_t \overset{iid}{\sim} \mathrm{N}(0, 1)$ and $a = 0.08$.

In case of kernel estimators the MSE value depends on the bandwidth b. In the kernel estimator we use the Epanechnikov kernel which is defined as $w(z) = \frac{3}{4}(1-z^2)$. Our bandwidth choices are b_opt1, b_opt2 and b_opt3. Algorithm of quantile estimation using kernel estimator is given by:

- Given X_1, X_2,...,X_n.
- Bandwidths chosen as mentioned above.
- Estimate the cumulative distribution function of the X_i's as $\hat{F}(x)$ using the kernel estimator.
- Using uniroot method, we find q as the solution of $\tilde{F}(q) = p$.

In the L-estimator we use the Epanechnikov kernel which is defined as $w(z) = \frac{3}{4}(1-z^2)$. Here we use our proposed bandwidth which we obtain by using Bootstrap method for estimating the quantile. Algorithm of quantile estimation using L-estimator is given by:

- Given X_1, X_2,...,X_n.

- Bandwidth estimated using Bootstrap method.
- Epanechnikov kernel is considered in the kernel quantile estimator.
- Quantile is estimated using the L-estimator SM_p.

In the GPD estimator, main issue is the choice of threshold k, tail parameter ξ and shape parameter β. There is a proposal given by Huisman et al. [20] about the choice of k which is given as $\frac{n}{2}$, where n is the sample size. But we choose k as $n-(\frac{n}{2})^p$, where n is the sample size and p is the probability, because $\frac{n}{k} \approx 1$. We estimate the $\hat{\xi}$ and $\hat{\beta}$ by fitting the generalized Pareto distribution (gPd) to the dataset using the asymptotic maximum likelihood technique. Algorithm of quantile estimation using GPD estimator is given by:

- Given X_1, X_2,. . .,X_n.
- k is chosen as $n-(\frac{n}{2})^p$, where n is the sample size and p is the probability.
- We estimate $\hat{\xi}$ and $\hat{\beta}$ by fitting generalized Pareto distribution (gPd) to the dataset using asymptotic maximum likelihood method.
- Considering the chosen k, $\hat{\xi}$ and $\hat{\beta}$, we obtain the quantile using the GPD estimator.

The results of the mean square error (MSE) of the quantile estimators at 99% are reported in Table 5.1 and 5.2 which are based on 500 simulations.

Comparing ES estimators

We compare the performance of five ES estimators viz: Emp_p, $\widehat{ES}_p$, $ES_{p,\beta}$, $ESker_p$ and Ja_p. As a measure of performance we use the MSE of each of the mentioned estimators. We ran simulations using five different underlying distribution functions such as Generalized Pareto distribution (GPD) with shape parameter $\xi = 0.3$, standard Normal, standard Lognormal, Weibull and Pareto(1,2) distributions. We also consider the following models in our simulation study:-

1. AR(1) model: $X_t = aX_{t-1} + \sqrt{1-a^2}Z_t$, $Z_t \overset{iid}{\sim} \mathrm{N}(0,\,1)$ and $a = 0.08$.
2. Log model: $log(X_t) = alog(X_{t-1}) + \sqrt{1-a^2}Z_t$, $Z_t \overset{iid}{\sim} \mathrm{N}(0,\,1)$ and $a = 0.08$.

The sample sizes considered are $n = 30$, 100, 250, 500, 1000 and 2500. Computations have been obtained by computing $m = 500$ samples from each distribution. The results of the mean square error (MSE) of the different estimators at 99% are reported in Table 5.3 and 5.4 which are based on 500 simulations.

From Table 5.1 and 5.2 we observe that no such quantile estimator outperforms the sample quantile. Also from Table 5.3 and 5.4 we observe that no such ES estimators outperforms the empirical estimator of ES. Based on our simulation study in Table 5 and 6 we have compared the MSE of ES and MS, using the empirical estimator. The ES and MS is estimated at 99%. MS at 99% means VaR at 99.5% We know that mean as a measure of central tendency has some disadvantages, it is highly sensitive to extreme values. But median is not affected by extreme values. So we try to consider median shortfall, which is nothing but the VaR at some other level greater than p. From Table 5.5 and 5.6 we observe that MS outperforms the ES for all the sample sizes.

Table 5.1: MSE estimates for different quantile estimators at 99%

$Dist^n$	n	SQ_p	A_p	B_p	SM_p	EV_p	$C-T_p$
N(0,1)	30	0.3021	0.2184	0.4524	1.2346	3.1943	0.2731
	100	0.1098	0.1079	0.1296	0.5184	1.3312	0.0941
	250	0.0466	0.0399	0.0493	0.2134	0.8602	0.0455
	500	0.0249	0.0227	0.0260	0.0725	0.6640	0.0212
	1000	0.0121	0.0115	0.0139	0.0155	0.5835	0.0117
	2500	0.0047	0.0043	0.0051	0.0057	0.5301	0.0045
Cauchy(0,1)	30	83453.04	83471.4	83506.04	197611.2	2843.124	52708.34
	100	103909.7	191934.4	191988.7	32376.8	508.3795	272921.6
	250	2000.355	2000.265	1999.861	919.5306	242.7881	983.2474
	500	393.8452	450.9489	450.1569	309.2897	122.3019	364.5123
	1000	123.0716	127.8621	126.6533	149.7986	73.7751	124.7571
	2500	37.7746	38.1889	38.0892	90.0211	44.1427	51.2886
Lognormal	30	24.0283	23.5209	22.3499	31.5986	46.6218	42.4659
	100	22.1430	43.7863	42.8084	16.1018	15.9679	19.4004
	250	5.0801	5.0616	4.9318	7.0553	6.6736	5.2996
	500	2.9080	3.0660	2.8873	2.9093	3.2179	2.5558
	1000	1.3373	1.3704	1.3360	1.4703	1.6222	1.3297
	2500	0.5046	0.5075	0.4890	0.5563	0.7259	0.5030
GPD	30	29543.91	29552.17	29563.12	34697.15	9802.844	199071.8
	100	94346.04	149446.6	148943.6	85121.86	4803.303	857537.9
	250	5540.409	5540.32	5539.736	6038.39	2528.732	6292.111
	500	3265.157	3797.957	3802.449	97614.15	1435.522	2713.922
	1000	1111.158	1073.496	1073.576	1055.028	834.8196	1024.88
	2500	409.2309	410.8855	411.9501	589.811	602.0519	420.2394
Weibull	30	179.8249	178.2669	174.4697	177.5919	227.031	117.4224
	100	161.9309	175.907	176.389	100.4599	84.6018	131.3887
	250	41.6090	41.5518	41.1869	51.1034	36.5847	30.3875
	500	19.1634	19.5869	19.4033	26.6500	20.4856	18.4952
	1000	9.6057	9.7702	9.5471	8.9786	11.5562	8.1685
	2500	4.0845	4.1515	4.0562	3.3065	8.8549	3.1510

Table 5.2: MSE estimates for different quantile estimators at 99% for dependency cases

Distribution	n	SQ_p	A_p	B_p	SM_p	EV_p	$C-T_p$
AR(1)	30	0.8797	0.7442	0.6660	2.3555	3.8995	0.7503
	100	0.2497	0.3163	0.3121	0.8671	1.6562	0.2671
	250	0.1363	0.1465	0.1419	0.3265	1.0337	0.1193
	500	0.0836	0.0828	0.0778	0.1258	0.7538	0.0792
	1000	0.0400	0.0398	0.0391	0.0362	0.6492	0.0447
	2500	0.0155	0.0151	0.0149	0.0157	0.5418	0.0140
Logmodel	30	63.3921	63.6665	64.8024	46.5482	68.6584	41.3238
	100	52.6390	61.2865	57.5780	25.6734	58.4928	35.8576
	250	24.1053	24.0814	23.9264	14.2551	19.3403	14.2426
	500	12.2285	11.9068	12.0002	7.8204	8.6404	8.3246
	1000	4.3711	4.3882	4.3281	3.5867	4.5402	5.2485
	2500	1.7293	1.7288	1.7211	1.6212	1.9388	1.4905

Table 5.3: MSE estimates for different estimators of ES at 99%

$Dist^n$	n	Emp_p	$\widehat{ES}_p$	$ESker_p$	Ja_p	$ES_{p,\beta}$
N(0,1)	30	0.8582	1.4433	10.3639	10.2223	2.7447
	100	0.0915	0.6825	10.3060	10.0816	0.8325
	250	0.1393	0.4230	10.3335	10.0704	1.0598
	500	0.0525	0.3199	10.3333	10.0360	0.4096
	1000	0.0642	0.2567	10.3398	10.0276	0.3468
	2500	0.1187	0.2107	10.3416	10.0418	0.3053
Lognormal	30	60.7349	119.1243	272.7406	392.8992	156.4774
	100	139.3741	68.3744	274.0812	386.275	69.6180
	250	26.0945	35.9818	274.7235	385.815	53.1004
	500	16.0188	23.9902	274.9052	384.7502	25.3831
	1000	10.0323	16.5716	274.9979	384.576	17.3348
	2500	7.9684	11.9737	275.0282	384.5493	12.0218
GPD($\xi = 0.3$)	30	83.0172	112.795	241.6518	336.8144	131.3027
	100	282.9646	81.7959	242.844	331.0379	81.2056
	250	46.9397	38.4159	242.8988	331.3232	52.1702
	500	24.5304	21.4161	243.1578	330.2429	23.124728
	1000	10.4128	12.6812	243.2042	330.1392	12.4728
	2500	5.1543	7.0818	243.2517	330.091	6.4008
Weibull	30	315.3055	418.2291	771.0513	1001.842	346.3367
	100	528.9571	217.5887	776.9215	981.0576	128.5861
	250	116.4806	81.9190	777.8451	981.708	80.1105
	500	71.2799	38.5271	777.9547	978.828	42.4353
	1000	30.5209	21.1050	778.355	978.2527	25.1131
	2500	12.5626	8.7534	778.208	978.429	14.8905
Pareto(1,2)	30	160.6576	195.7786	308.7192	464.0539	212.6719
	100	253.623	109.8344	311.3657	455.067	109.5845
	250	72.3071	67.6917	311.7332	454.8516	82.3083
	500	69.7520	49.3579	311.8147	453.8522	56.3831
	1000	38.6420	30.1201	311.9519	453.6058	35.5263
	2500	19.7902	15.1763	311.9264	453.617	18.6157

Table 5.4: MSE estimates for different estimators of ES at 99% for dependency case

$Dist^n$	n	Emp_p	$\widehat{ES}_p$	$ESker_p$	Ja_p	$ES_{p,\beta}$
AR(1)$a = 0.08$	30	0.9134	1.6618	3.8391	10.2154	2.7452
	100	2.7173	0.8318	2.6944	10.0879	0.8344
	250	0.1075	0.5563	2.6654	10.0715	1.0675
	500	0.0580	0.4334	2.2207	10.0381	0.4128
	1000	0.1194	0.3665	2.3129	10.0273	0.3480
	2500	0.2023	0.3175	2.2797	10.0419	0.3059
Logmodel $a = 0.08$	30	92.9383	134.9161	284.3368	390.6363	156.4489
	100	133.9765	70.0918	273.754	387.5749	69.9788
	250	28.2944	42.0117	273.4392	385.8003	55.3510
	500	16.0799	27.6974	268.7672	384.9032	26.0661
	1000	10.7926	19.5709	269.7539	384.5136	17.5372
	2500	8.9290	13.6947	269.3914	384.4788	12.0309

Table 5.5: MSE estimated using empirical estimator of ES and MS

Distribution	n	ES at 99%(empirical)	VaR at 99.5%(empirical)
N(0,1)	30	0.8582	0.5684
	100	3.0915	0.1984
	250	0.1393	0.0780
	500	0.0525	0.0505
	1000	0.0642	0.0223
	2500	0.1187	0.0088
Lognormal	30	60.7349	56.4495
	100	139.3741	73.7233
	250	26.0645	15.1754
	500	16.0188	10.9944
	1000	10.0323	4.2147
	2500	7.9684	1.6079
GPD($\xi = 0.3$)	30	83.0172	59.3554
	100	282.9646	190.8207
	250	46.9397	24.1594
	500	24.5304	9.8929
	1000	10.4128	4.2903
	2500	5.1542	1.8361
Weibull	30	315.3055	261.652
	100	528.9571	222.8295
	250	116.4806	75.8402
	500	71.2799	51.1641
	1000	30.5209	22.3390
	2500	12.5626	8.1194
Pareto(1,2)	30	160.6576	195.4856
	100	253.623	232.422
	250	72.3071	58.2088
	500	69.7520	31.5482
	1000	38.6420	11.3928
	2500	19.7902	3.6221

Table 5.6: MSE estimated using empirical estimator of ES and MS for dependency case

Distribution	n	ES at 99%(empirical)	VaR at 99.5%(empirical)
AR(1)$a = 0.08$	30	0.9134	0.5280
	100	2.7173	0.1988
	250	0.1075	0.0803
	500	0.0580	0.0425
	1000	0.1194	0.0201
	2500	0.2023	0.0091
Logmodel $a = 0.08$	30	92.9383	53.2411
	100	133.9765	82.6992
	250	28.2944	16.2666
	500	16.0799	8.2956
	1000	10.7926	3.6681
	2500	8.9290	1.6351

6 Conclusion

In this paper we have discussed about the estimators of VaR, ES and MS and compared the performance of the estimators using the Monte Carlo simulations. From the simulation study it is observed that no such estimator is seen to outperform the empirical estimators of VaR, MS and ES. Hence we use the empirical estimators of ES and MS to compare its performance. We observe that MS outperforms the ES for all the sample sizes which is considered in our simulation study. Hence from our simulation study we can say that MS is a more preferable risk measure than ES, where MS is also a coherent risk measure.

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A Survey on Some Examples and Computations in Supermodular Games

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Abstract. In this article we will discuss about the games of monopoly, surplus sharing, bankruptcy and airport along with their supermodularity. In later part, we will discuss the strong and weak least cores of a supermodular game using the concept of the theory of minimising crossing submodular functions. Further we use it to determine the strong and weak least core of induced subgraph games.

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1 Introduction

Let us go through some basic definitions and results to be used in the later part of our study. Throughout the study, set of all n players will be denoted by $N = \{1, 2, ..., n\}$.

Definition 1.1. *[2] A cooperative game in characteristic function form is an ordered pair (N, v), where $N = \{1, 2, ..., n\}$ is the set of all players and function $v : 2^N \to \boldsymbol{R}$ is the characteristic function which assigns each subset (coalition) of N a real value.*

For each coalition $S \subseteq N$, the value $v(S)$ is called worth of the coalition or coalitional value for S.

Definition 1.2. *[2] In a cooperative game (N, v), for any subset $M \subseteq N$, the game (M, v) involving only the players of coalition M with respect to the same characteristic function v is called as **subgame** of (N, v).*

Definition 1.3. *[2] A cooperative game (N, v) is said to be **super-additive** if $v(S)+v(T) \leq v(S \cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Definition 1.4. *[2] A cooperative game (N, v) is said to be **sub-additive** if $v(S)+v(T) \geq v(S \cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Equivalently a cooperative game (N, v) is said super-additive if $(N, -v)$ is sub-additive and vice-versa.

Definition 1.5. *[2] A cooperative game (N, v) is said to be **additive** if $v(S) + v(T) = v(S \cup T)$ for all coalitions S and T of N such that $S \cap T = \phi$.*

Definition 1.6. *[2] In a cooperative game (N, v), **marginal contribution** of player $i \in N$ with respect to a coalition S is the the value $v(S \cup \{i\}) - v(S)$, where $i \in N \setminus S$*

Definition 1.7. *[8] A cooperative game (N, v) is said to be **monotonic** if $v(S) \leq v(T)$ whenever $S \subseteq T \subseteq N$.*

If the players in set N with respect to the game (N, v) decide to work together, then a natural question arises as how to distribute the coalitional value among them so that the division is fair enough to everyone. In such cases we go for a solution vector as defined below.

Definition 1.8. *[8] For a cooperative game (N, v), an **allocation vector** or a **payoff vector** is an n-coordinated vector $x = (x_1, x_2, ..., x_n)$,*

x_i in the payoff vector above is the amount received by player i. Further for any coalition $S \subseteq N, x(S)$ is the sum of the payoffs received by players of the coalition S. That is $x(S) = \sum_{i \in S} x_i$.

Definition 1.9. *[8] In a cooperative game (N, v), a payoff vector x is said to be **individually rational** if $x_i \geq v(\{i\})$.*

Definition 1.10. *[2] In a cooperative game (N, v), a payoff vector x is said to be **collective rational** if $x(S) \geq \sum_{i \in S} x_i$ for all $S \subseteq N$.*

Definition 1.11. *[2] In a cooperative game (N, v), a payoff vector x is said to be **totally rational** or **pareto efficient** if $x(S) = v(N)$.*

In a cooperative game (N, v), **pre-imputation** [2] set is the collection of all pareto payoff vectors, further **imputation** is the set of all pareto and totally rational vectors. Now we are in a position to introduce the concept of core which will be an important tool to determine the supermodularity of a game.

Definition 1.12. *[2] In a cooperative game (N, v), a payoff vector x is said to be in the **core**, if x is totally and collective rational. Further the collection of all such vectors is called core of the game and is denoted by $c(v)$. That is $c(v) = \{x \in \boldsymbol{R}^n : x(S) \geq v(S), x(N) = v(N); S \subseteq N\}$*

[2] The Shapley value is an interesting solution concept in a cooperative game. Choosing a particular solution concepts becomes an ambiguous work as it may not seem reasonable to many players. Shapley [2] characterised a unique solution using a collection of intuitively reasonable axioms.

Definition 1.13. *[2] In a cooperative game (N, v), the **Shapley Value** ϕ is the solution $(\phi_1^{sh}, \phi_2^{sh}, ..., \phi_n^{sh})$, where $\phi_i^{sh} = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}[v(S \cup \{i\}) - v(S)]$ for all coalitions S in N.*

Another way to calculate the Shapley value is by using the permutations of the player set N as average of the marginal vectors of the game. That is $\phi_i^{sh} = \frac{1}{|N|!} \sum_{\pi \in \Pi} m^{\pi}$, where Π is the collection of all permutations of N and m^{π} is the marginal contribution of player$i$ with respect to all coalitions. It is to be noted here that the Shapley value is unique.

Definition 1.14. *[4] Two sets S and T are said to be **crossing** in N if $S \cap T \neq \phi, S \backslash T \neq \phi, T \backslash S \neq \phi, S \cup T \neq N$. Further a function $f : 2^N \to R$ is called **crossing submodular** if $f(S) + f(T) \geq f(S \cap T) + f(S \cup T)$ holds for all subset S and T of N.*

Definition 1.15. *[4] The family of sets S called a **copartition** of the set of players N if $\{N \backslash S; S \in S\}$. Further $P_k(N)$ is the collection of all the partitions S of N having at least k sets and $\overline{P_k}(N)$ is the collection of all the copartitions S of N having at least k sets.*

Definition 1.16. *[4] A **hypergraph** G is defined as a pair (N, E), where N is the set of vertices and E is the set of hyperedges between vertices. Each **hyperedge** E is the set, $E \subseteq N$.*

Definition 1.17. *[1] For any two elements x and y in a poset S, $x \vee y$ **join** and $x \wedge y$ **meet** are defined as the supremum and infimum of x and y respectively.*

2 Examples of Supermodular Games

Here, we will discuss about the supermodularity of some popular games like: monopoly, bankruptcy, airport and surplus sharing.

The Monopoly Firm Game

[7] Consider a firm which manufactures non-negative quantity of m products $j = 1, 2, ..., m$.Let x be an m vector which denotes production of the m products. That is $x = (x_1, x_2, ..., x_m)$, where x_i denotes the production of i^{th} product and further let $c(x)$ denote the total cost of the firm for the production vector x.

We consider the firm to be a monopoly with its monopoly production level y, where y is an m vector as defined in the case of x above. Let us consider that the firm has n consumers and let $N = \{1, 2, ..., n\}$ be the set of consumers/players. Given that market price of the product would yield a total demand equal to the monopoly production level y. Let the consumption of each player i be the non-negative vector $x^(i)$and $\sum_{i \in N} x^(i) = y$, where in $x^(i) = (x_1^i, x_2^i, ..., x_m^i)$, x_j^i denotes the product x^j consumed by player i.

The firm sets prices for its m products such that r_i is the revenue received from vector $x^(i)$ demand by the player i. The firm wants to find the n-vector r of revenues from n players such that the total revenue generated is equal to the total cost.

That is $\sum_{i \in N} = c(y)$.

Further the firm expects no rival firms to produce the demands for consumers amongst the n consumers for less than the total revenue obtained from those consumers by the firm.

That is $\sum_{i \in S} r_i \leq c(\sum_{i \in S} x^i)$, where $S \subseteq N$ us a subset of consumers N.

Now define $v : 2^N \to \mathbf{R}$ by $v(S) = -c(\sum_{i \in S} x^i)$ for each subset S of N.

Then the game (N, v) is called the game of monopoly firm.

Consider a set S containing n tuples and $c(x)$ be the cost function corresponding to the vectors of S. Then the cost function is said to exhibit **weak cost complementary** if $c(x + y) - c(x) \geq c(x + y + z) - c(x + z)$ for all vectors $x, y, z \in S$ such that $y_i > 0$ implies $z_i = 0$.

Theorem 2.1. *Consider the cooperative game (N, v) of the monopoly firm.*
(a)[7] If $x^{(i)} \wedge x^{(j)} = 0$ for each pair of distinct consumers i and j and if the production cost function $c(x)$ exhibits weak cost complementarity then the cooperative game (N, v) is a supermodular game.

(b)[9] If the production cost function $c(x)$ exhibits cost complementarity, then the cooperative game (N, v) is a supermodular game.

The Bankruptcy Game

[3] Bankruptcy problems arise when a number of individuals claim on a particular resource whose total worth is insufficient to meet the claims of the individuals. For example we can consider the property of a dying man who owed from a lot of person. But if we divide the property amongst the claimants it turns out to be sufficiently less to the claims of the claimants.

Mathematically a bankruptcy problem is written as an ordered pair (E, d) in $R \times R^n$ is he total value of the estate, $0 \leq d_1 \leq d_2 \leq ... \leq d_n$ is such that $0 \leq E \leq d_1 + d_2 + ... + d_n = D$. The d_i^s above are claims made by players i from the set of players $N = 1, 2, ..., n$.

There are different ways to divide the value E among the players assigning each claimant a fair and a justified amount. But here we shall refrain ourself from the ways of division and particularly focus on the supermodularity of the bankruptcy game.

For any bankruptcy problem (E, d), a bankruptcy game $v_{E,d} : 2^N \to R$ by $v_{E,d}(S) = (E - d(N - S)) = max\{E - d(N - S), 0\}$ for all $S \subseteq N$. In other words $v_{E,d}$ is that worth of the estate which is not claimed by the complement of S.

Theorem 2.2. *[3] Bankruptcy game is a supermodular game.*

The Surplus Sharing Game

[10] Here we consider n number of players/agents operating either independently or cooperatively forming coalitions including the grand coalition. Let $N = \{1, 2, ..., n\}$ be the set of all players/agents. Further let us suppose r_i to be the return when player i acts independently and q to be the return when all players act cooperatively and form the grand coalition. Assume that $q > \sum_{i \in N} r_i$, thus n agents acting cooperatively receive a joint surplus of $q - \sum_{i \in N} r_i > 0$ compared to what they would have received in case they operate independently.

Surplus sharing problem basically involves in finding a payoff vector $x = (x_1, x_2, ..., x_n)$, where x_i is the return given to each player i such that x_i is greater to the return that player i could have earned working independently. That is $y_i \geq r_i$ for eery player i. Further sum of the distributions of all players is equal to the return by n agents contributing cooperatively. That is $q = \sum_{i \in N} x_i$.

Theorem 2.3. *[10] For each coalition S, define $v : 2^N \to R$ by $v(S) = \sum_{i \in S} r_i$ if $S \neq N$ and $v(N) = q$, then the game (N, v) is a supermodular game if and only if $q \geq \sum_{i \in N} r_i$.*

It may be noted here that x is a solution for the for the surplus sharing problem if and only if x is in the core of the game (N, v).

The Airport Games

[6] Here we discuss on two games related to an airport. In particular "Airport Cost Game" and "Airport Profit Game ". The characteristic function of the former game deals with the cost required to construct a runway for different flights that land or take off at an airport whereas the characteristic function of the latter game deals with the profit earned by the airport authority by allowing flights to avail the runway of the airport.

Let $N = \{1, 2, ..., n\}$ be the set of all flights(players) that might use the runway of the airport. Let r_i be the revenue earned by airport authority from flight i and c_i be the cost required to construct the runway for flight i. Consider runway length to be the only salient feature for a flight to be used in the airport.Without the loss of generality let us consider the players of $N = \{1, 2, ..., n\}$ such that $c_1 \leq c_2 \leq ... \leq c_n$.That is, we consider the sequence of the players in N in ascending order of their costs for preparing the corresponding runway.

If c_i and c_j are costs corresponding to flights i and j respectively such that $c_i \leq c_j$, then the runway prepared for flight i is sufficient enough for flight j.

Airport Cost Game [6] For set $N = \{1, 2, ..., n\}$ of flights, define $v : 2^N \longrightarrow R$ by

$v(S) = max_{i \in S} c_i$. In other words game v is the maximum cost required by a flight in S.

Airport Profit Game [6] Airport profit game is where we get profit earned by an airport by allowing a set of flights to use the runway.

So $v : 2^N \longrightarrow R$ defined as

$v(S) = \sum_{i \in S} r_i - max_{i \in S} c_i$ is known as the airport profit game.

Theorem 2.4. *[5] The Airport Profit Game (N, v) is supermodular.*

3 Computations in Supermodular Games

Here we will study the strong and weak least cores of a supermodular game using the theory of minimising crossing submodular functions. Further we use the formula to determine the strong and weak least core values of the induced subgraph game.

Strong and Weak Least Core of Supermodular Games

As the core of a game consists of payoff vectors which are individually rational, pareto efficient and collective rational, let us now consider the following definitions for further extension of the core concept.

Definition 3.1. *[4] For $\varepsilon \in R$, a payoff vector $x \in R^n$ is said to be in the **strong** $(-\varepsilon)-$ **core** if the loss of a deviating nontrivial coalition is at least ε.That is $x(S) - v(S) \geq \varepsilon$, for any coalition $S \subseteq N$.*

Definition 3.2. *[4] For $\varepsilon \in R$,a payoff vector $x \in R^n$ is said to be in the **weak** $(-\varepsilon)-$ **core** if the loss of a deviating nontrivial coalition is at least ε on average. That is $x(S) - v(S) \geq \varepsilon|S|$, for any coalition $S \subseteq N$.*

By relating the definitions given above to the definition of the core of a game we note that a core of a game is simply the strong 0-core and the weak 0-core.

Definition 3.3. *[4] The minimum $\varepsilon \in R$ for which the strong ε-core is nonempty is called as **strong least core value** of the game. Further the **strong least core** of a game is the ε-core for the corresponding strong least core value ε.*

Definition 3.4. *[4] The minimum $\varepsilon \in R$ for which the weak ε-core is nonempty is called as **weak least core value** of the game. Further the **weak least core** of a game is the ε-core for the corresponding weak least core value ε.*

As the core of a supermodular game is nonempty[] so the strong and weak least core values of a supermodular game defined above is always nonpositive. Therefore it is convenient to write strong and weak $-\varepsilon$-cores instead of strong and weak ε-cores. Throughout the chapter we consider **strong and weak $-\varepsilon$-core instead of strong and weak ε-core**.

Let $\delta^s(\varepsilon, S) = \varepsilon$ and $\delta^s(\varepsilon, S) = \varepsilon|S|$. Then for all $i \in N$ and $S \subseteq N$ the strong and weak least core values of a game can be computed by solving

$$\begin{aligned} x(S) \geq & v(S) + \delta^{type}(\varepsilon, S) \\ x(N) = & v(N) \\ x(v) \geq & 0 \end{aligned}$$

where δ^{type} represents δ^s in case of strong core and δ^w in case of weak core.
Now let us fix $\varepsilon \geq 0$ and define a function $f_\varepsilon^{type} : 2^N \to R$ by

$$f_\varepsilon^{type}(S) = \begin{cases} 0 \quad if\ S = \phi, \\ -v(S) - \delta^{type}(\varepsilon, S) \quad if \phi \subsetneq S \subsetneq N, \\ -v(N) \quad if S = N \end{cases}$$

Clearly the function defined above is crossing submodular.

Definition 3.5. *[4] For any function $f : 2^N \to R$, define **extended polymatroid** associated with f as $P(f) = \{x \in R^n | x(S) \leq f(S); S \subseteq N\}$. Further $p_\varepsilon^{type}(S) = max\{x(N) | x \in P(f_\varepsilon^{type})\}$.*

We now present two theorems for computing strong and weak least cores of a supermodular game.

Theorem 3.6. *[4] The strong least core value of a supermodular game (N, v) is*

$$-min\{min_{S \in P_2(N)} \frac{1}{|S|}(v(N) - \sum_{S \in S} v(S)), min_{S \in \bar{P}_2(N)} \frac{1}{|S|}((|S| - 1)v(N) - \sum_{S \in S} v(S))\}$$

.

Theorem 3.7. *[4] The weak least core value of a supermodular game game (N, v) is $-\frac{1}{|N|} min_{S \in \bar{P}_2(N)}(v(N) - \frac{1}{(|S|-1)} \sum_{S \in S} v(S))$.*

Computation of Least Core in Induced Subgraph Game

Let $N = \{1, 2, ..., n\}$ be the set of n vertices and $G = (N, E, w)$ be a weighted hypergraph, where E is the set of hyperedges and $w : E \to R$ is a weight function which assigns every hyperedge e a weight in R. In case of hypergraph, hyperedge e is a subset of the set of vertices N, that is $e \subseteq N$.

Definition 3.8. *[4] For any weighted hypergraph $G = (N, E, w)$,the **induced subgraph game** is the cooperative game (N, v), where $v : 2^N \to R$ is the total weight of the hyperedges $e \in E$ with $e \subseteq S$.That is $v(S) = \sum_{e \subseteq S} w(e)$.*

Being the sum of the weights of the hyperedges clearly the game (N, v) is supermodular.

Definition 3.9. *[4] Let $S \subseteq N$ be a set of vertices, then we say that a hyperedge e is **cut** by coalition S if $e \nsubseteq S$ and $e \nsubseteq N \setminus S$. The total weight of hyperedges e cut by a coalition S is called as the **cut weight** of S and is denoted by $c(S)$. A **minimum cut** is the minimum cut weight among all possible coalitions of N and is denoted by $c^*(G)$, where $G = (N, E, w)$ is the associated weighted hypergraph.*

Definition 3.10. *[4] A hyperedge e is said to be **cut** by a partiton S of N if $e \nsubseteq S$ for all $S \subseteq \mathrm{S}$. The total weight of hyperedges cut by a partiotion S of N is called **cut weight** of the partition S denoted by $c(\mathrm{S})$. That is $c(\mathrm{S}) = \sum_{e \in E; e \nsubseteq S \in \mathrm{S}} w(e)$. Further $\bar{c}(G) = \sum_{S \in \mathrm{S}} c(S) - c(\mathrm{S})$.*

Theorem 3.11. *[4] The strong and weak least core value of the induced subgraph game associated with a hypergraph $G = (N, E, w)$ is $-min_{\mathrm{S} \in P_2(N)} \frac{c(\mathrm{S})}{|\mathrm{S}|}$.*

Theorem 3.12. *[4] The strong and weak least core value of the induced subgraph game associated with a hypergraph $G = (N, E, w)$ is $-min_{\mathrm{S} \in P_2(N)} \frac{c(\mathrm{S})}{|\mathrm{S}|}$.*

Theorem 3.13. *[4] The strong least core value of the induced subgraph game associated with the subgraph $G = (N, E, w)$ is at least $-\frac{c^*(G)}{2}$.*

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